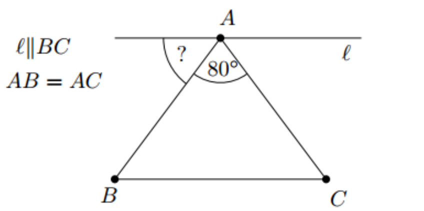
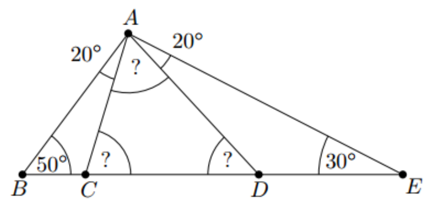
Geometry 1 Problem Set

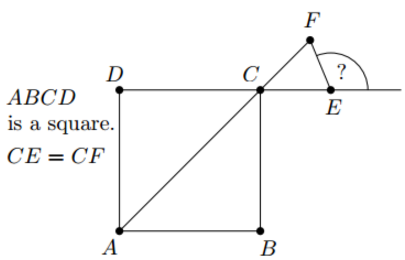
Math Circle Competition Team

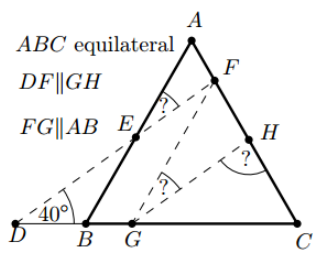
1 Introduction to Geometry

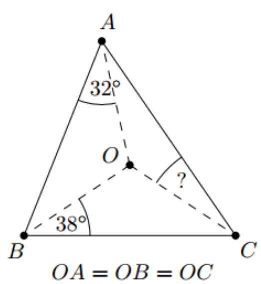
1. In the following problems, find the angles denoted by question marks.

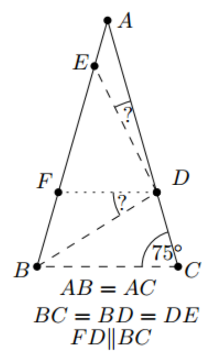






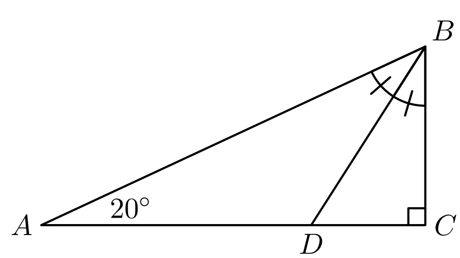






2. [AMC 10A 2012] Let $\angle ABC = 24°$ and $\angle ABD = 20°$. What is the smallest possible degree measure for $\angle CBD$?

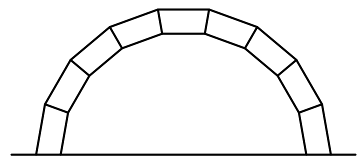
3. [AHSME 1986] $\triangle ABC$ is a right angle at C and $\angle A = 20°$. If BD is the bisector of $\angle ABC$, then what is $\angle BDC$?



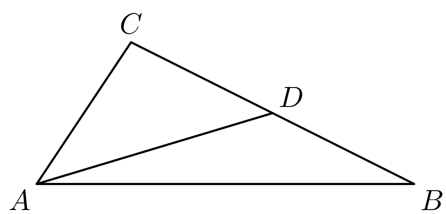
4. In $\triangle ABC$, suppose there exists a point X on the side $\overline{BC}$ such that $XA = XB = XC$. Show that $\angle BAC = 90°$.

5. [Math League HS 2013-2014/2009-2010/1994-1995] In a certain quadrilateral, the three shortest sides are congruent, and both diagonals are as long as the longest side. What is the degree measure of the largest angle of this quadrilateral?

6. [AMC 10B 2009] The keystone arch is an ancient architectural feature. It is composed of congruent isosceles trapezoids fitted together along the non-parallel sides, as shown. The bottom sides of the two end trapezoids are horizontal. In an arch made with 9 trapezoids, let x be the angle measure in degrees of the larger interior angle of the trapezoid. What is x?

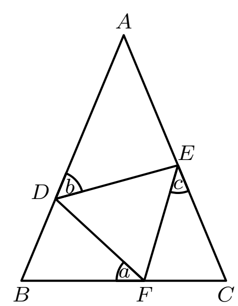


7. [AHSME 1957] In triangle ABC, $AC = CD$ and $\angle CAB - \angle ABC = 30°$. What is $\angle BAD$?



8. In quadrilateral $ABCD$, $AB = BC$, $\angle ABD = 30°$, $\angle C = 50°$, and $\angle CBD = 80°$. Find the measure of $\angle A$.

9. [AHSME 1960] In this diagram AB and AC are the equal sides of an isosceles triangle ABC, in which is inscribed equilateral triangle DEF. Designate angle BFD by a, angle ADE by b, and angle FEC by c. Then:



(A) $b = \frac{a+c}{2}$ **(B)** $b = \frac{a-c}{2}$ **(C)** $a = \frac{b-c}{2}$
(D) $a = \frac{b+c}{2}$ **(E)** none of these

10. [MAΘ 1992] In regular polygon $ABCDE\cdots$, we have $\angle ACD = 120°$. How many sides does the polygon have?

11. Let ABC be an isosceles triangle with $AB = AC$. Suppose that there exist points D_1, D_2 on $\overline{AB}$ and E_1, E_2, E_3 on $\overline{AC}$ satisfying

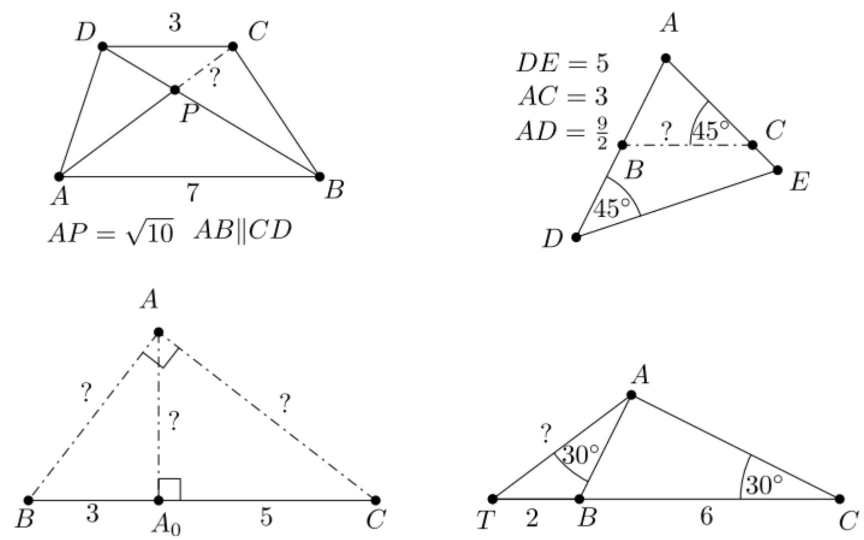
$$AE_1 = E_1D_1 = D_1E_2 = E_2D_2 = D_2E_3 = E_3B = BC.$$

Compute the measure of $\angle BAC$.

★ 12. [AMC 10B 2008] Quadrilateral $ABCD$ has $AB = BC = CD$, $\angle ABC = 70°$, and $\angle BCD = 170°$. What is the degree measure of $\angle BAD$?

2 Similarity

1. In the figures below, find the lengths of the segments denoted by question marks.

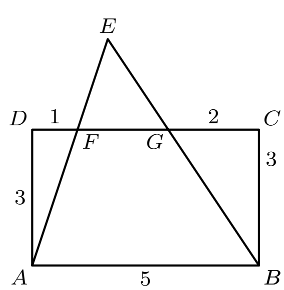


2. [CMIMC 2017] Let ABC be a triangle with $\angle BAC = 117°$. The angle bisector of $\angle ABC$ intersects side AC at D. Suppose $\triangle ABD \sim \triangle ACB$. Compute the measure of $\angle ABC$, in degrees.

3. [HMMT Geometry 2002] Let $\triangle ABC$ be equilateral, and let D, E, and F be points on sides BC, CA, AB respectively, with $FA = 9$, $AE = EC = 6$, $CD = 4$. Determine the measure (in degrees) of $\angle DEF$.

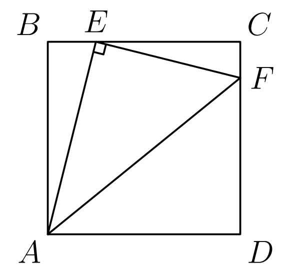
4. Let $\triangle ABC$ be a triangle, and let points D and E lie on $\overline{AB}$ and $\overline{AC}$ respectively. Show that $DE \parallel BC$ if and only if
$$\frac{AD}{AB} = \frac{AE}{AC}.$$

5. [AMC 12B 2003] In rectangle $ABCD$, $AB = 5$ and $BC = 3$. Points F and G are on $\overline{CD}$ so that $DF = 1$ and $GC = 2$. Lines AF and BG intersect at E. Find the area of $\triangle AEB$.



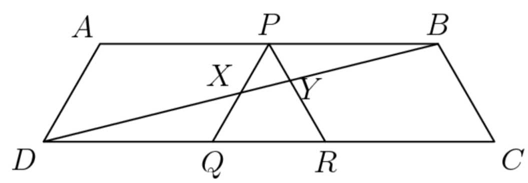
HINT: Drop altitudes!

6. [MATHCOUNTS 2015] Triangle AEF is a right triangle with $AE = 4$ and $EF = 3$. The triangle is inscribed inside square $ABCD$ as shown. What is the area of the square?



7. [CMIMC 2017] Triangle ABC has an obtuse angle at $\angle A$. Points D and E are placed on $\overline{BC}$ in the order B, D, E, C such that $\angle BAD = \angle BCA$ and $\angle CAE = \angle CBA$. If $AB = 10$, $AC = 11$, and $DE = 4$, determine BC.

8. [Mandelbrot] Figure $ABCD$ below has sides $AB = 6, CD = 8, BC = DA = 2$, and $AB \parallel CD$. Segments are drawn from the midpoint P of AB to points Q and R on side CD so that PQ and PR are parallel to AD and BC as shown. Diagonal DB intersects PQ at X and PR at Y. Evaluate PX/YR.



9. [AMC 10A 2016] In rectangle $ABCD$, $AB = 6$ and $BC = 3$. Point E between B and C, and point F between E and C are such that $BE = EF = FC$. Segments $\overline{AE}$ and $\overline{AF}$ intersect $\overline{BD}$ at P and Q, respectively. The ratio $BP : PQ : QD$ can be written as $r : s : t$, where the greatest common factor of r, s and t is 1. What is $r + s + t$?

   HINT: Extend AE and AF to intersect CD at X and Y respectively. Then scout for similar triangles.

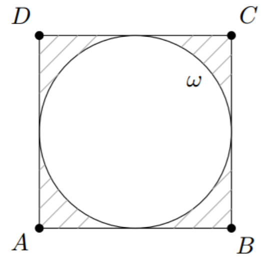
★ 10. [AIME 1998] Let $ABCD$ be a parallelogram. Extend $\overline{DA}$ through A to a point P, and let $\overline{PC}$ meet $\overline{AB}$ at Q and $\overline{DB}$ at R. Given that $PQ = 735$ and $QR = 112$, find RC.

★ 11. [Thomas Mildorf] ABC is an isosceles triangle with base $\overline{AB}$. D is a point on $\overline{AC}$ and E is the point on the extension of $\overline{BD}$ past D such that $\angle BAE$ is right. If $BD = 15, DE = 2$, and $BC = 16$, then compute CD.

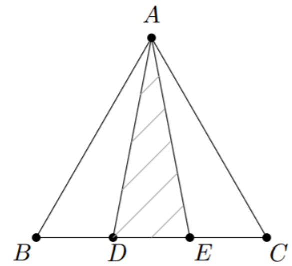
★ 12. [AIME 2003] In $\triangle ABC$, $AB = 360$, $BC = 507$, and $CA = 780$. Let M be the midpoint of $\overline{CA}$, and let D be the point on $\overline{CA}$ such that $\overline{BD}$ bisects angle ABC. Let F be the point on $\overline{BC}$ such that $\overline{DF} \perp \overline{BD}$. Suppose that $\overline{DF}$ meets $\overline{BM}$ at E. Find the ratio $DE : EF$.

3 Finding Areas

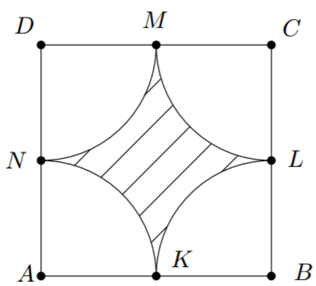
1. Compute the shaded areas below.



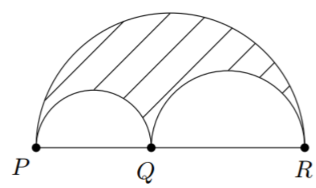
$ABCD$ square with $AB = 1$
ω incircle



ABC equilateral, $AB = 3$
$BD = DE = EC$



$ABCD$ square with $AB = 2$
K, L, M, N midpoints of sides
A, B, C, D centers of the arcs



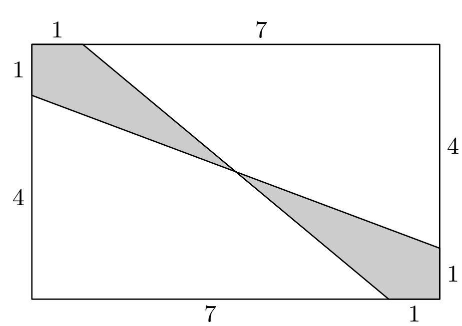
PQ, QR, PR are diameters
$PQ = 3$ $QR = 4$

2. Let $\triangle ABC$ be an acute triangle, and let F and F be the feet of the perpendiculars from B to AC and from C to AB, respectively. Suppose $AB = 12$, $AC = 18$, and $BE = 7$. What is CF?

3. [NIMO 28] Trapezoid $ABCD$ is an isosceles trapezoid with $AD = BC$. Point P is the intersection of diagonals AC and BD. If the area of $\triangle ABP$ is 50 and the area of $\triangle CDP$ is 72, what is the area of the trapezoid?

   HINT: Compute the area of $\triangle BCP$. To do this, first find the ratio $AP : PC$, and use this to compare the areas of $\triangle ABP$ and $\triangle BCP$.

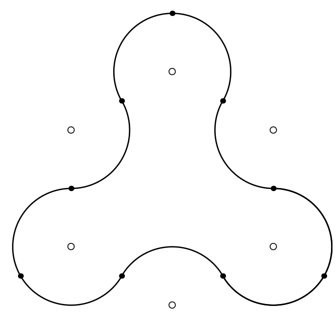
4. [AMC 10A 2016] What is the area of the shaded region of the given 8×5 rectangle?



5. [Mandelbrot 2006-2007] Suppose that $ABCD$ is a trapezoid in which $\overline{AD} \parallel \overline{BC}$. Given $\overline{AC} \perp \overline{CD}$, $\overline{AC}$ bisects angle $\angle BAD$, and $area(ABCD) = 42$, then compute $area(ACD)$.

6. [AMC 10A 2017] Sides $\overline{AB}$ and $\overline{AC}$ of equilateral triangle ABC are tangent to a circle as points B and C respectively. What fraction of the area of $\triangle ABC$ lies outside the circle?

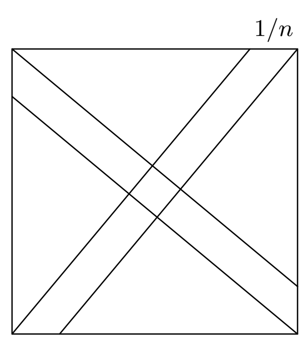
7. [AMC 10A 2012] The closed curve in the figure is made up of 9 congruent circular arcs each of length $\frac{2\pi}{3}$, where each of the centers of the corresponding circles is among the vertices of a regular hexagon of side 2. What is the area enclosed by the curve?



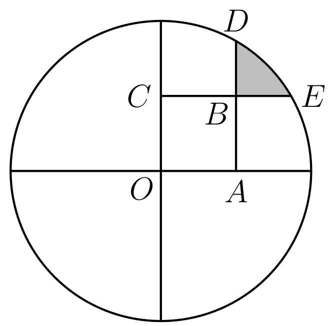
8. [AMC 10A 2013] Two sides of a triangle have lengths 10 and 15. The length of the altitude to the third side is the average of the lengths of the altitudes to the two given sides. How long is the third side?

9. [AIME 2001] In triangle ABC, angles A and B measure 60 degrees and 45 degrees, respectively. The bisector of angle A intersects $\overline{BC}$ at T, and $AT = 24$. Find the area of triangle ABC.

10. [AIME 1985] A small square is constructed inside a square of area 1 by dividing each side of the unit square into n equal parts, and then connecting the vertices to the division points closest to the opposite vertices. Find the value of n if the the area of the small square is exactly $1/1985$.



11. [AMC 10B 2006] A circle of radius 2 is centered at O. Square $OABC$ has side length 1. Sides $\overline{AB}$ and $\overline{CB}$ are extended past B to meet the circle at D and E, respectively. What is the area of the shaded region in the figure, which is bounded by $\overline{BD}$, $\overline{BE}$, and the minor arc connecting D and E?



★ 12. [AMC 10A 2013] A unit square is rotated 45° about its center. What is the area of the region swept out by the interior of the square?

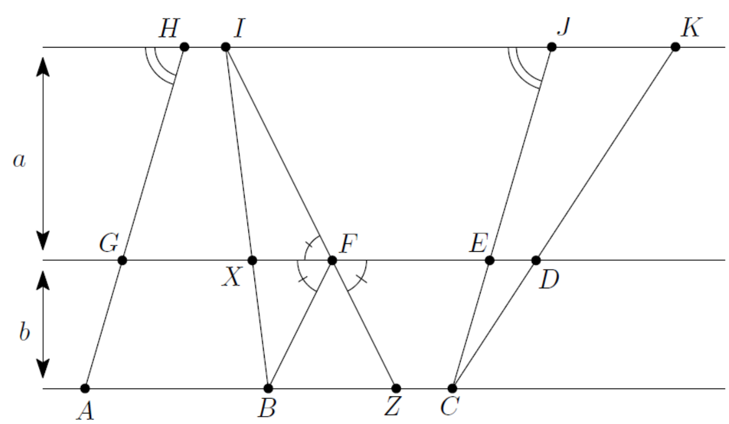
★ 13. [AMC 10A 2010] Equiangular hexagon $ABCDEF$ has side lengths $AB = CD = EF = 1$ and $BC = DE = FA = r$. The area of $\triangle ACE$ is 70% of the area of the hexagon. What is the sum of all possible values of r?

4 Levels, Menelaus, and Angle Bisectors

Note: Since we do not have a large repository of levels problems, some of the questions below will test Menelaus and the Angle Bisector Theorem.

1. Consider the diagram shown below. Assume the three horizontal lines are parallel to each other. Using only the information given, which of the following statements are true?

(a): $AG/AH = b/(a+b)$	(b): $BF = FZ$	(c): $IF/FB = a/b$
(d): $CE/EJ = CD/DK$	(e): $IF/BX = a/b$	(f): $JE/JC = a/b$
(g): $CE/DK = b/a$	(h): $ED/JK = b/a$	(i): $FI/FB = XI/XB$



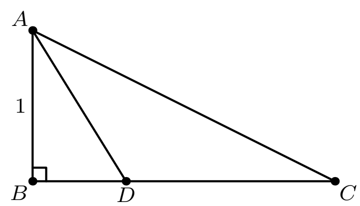
2. In isosceles triangle ABC, $AB = AC$. Let K and L be points on $\overline{AB}$ and $\overline{AC}$, respectively, such that $BK = \frac{1}{3}CL$ and $KL = 6$. Let P be the intersection of $\overline{KL}$ and the line BC. Find PK.

3. [OMO 2013] Points M, N, P are selected on sides $\overline{AB}$, $\overline{AC}$, $\overline{BC}$, respectively, of triangle ABC. Find the area of triangle MNP given that $AM = MB = BP = 15$ and $AN = NC = CP = 25$.

4. Let ABC be a triangle with point K on AB such that $BK/AB = 1/3$ and point L on AC such that $CL/AC = 1/4$. Let M be the midpoint of $\overline{KL}$ and N the intersection of AM and BC. Find NM/NA.

5. In acute $\triangle ABC$ points K and L are given on the side $\overline{AB}$ such that $AK = KL = LB$ and points M and N are given on the side $\overline{AC}$ such that $AM = MN = NC$. Show that $\overline{LN}$ bisects $\overline{KC}$.

6. [AMC 10B 2009] Triangle ABC has a right angle at B, $AB = 1$, and $BC = 2$. The bisector of $\angle BAC$ meets $\overline{BC}$ at D. What is BD?



7. Let ABC be a scalene triangle and denote by D the intersection of the angle bisector at A with line BC. Prove that
$$\frac{DB}{DC} = \frac{AB}{AC}.$$

8. Let ABC be a triangle with $AB < AC$. Points E and F are placed on $\overline{AC}$ and $\overline{AB}$ respectively such that $\angle ABE = \angle EBC$ and $\angle ACF = \angle FCB$. Line EF intersects BC at a point P. Suppose $PB = 13$ and $BC = 6$. Compute the ratio $\frac{AB}{AC}$.

9. Let M be the midpoint of the side BC of a triangle ABC. Point K on the segment AM satisfies $CK = AB$. Denote by L the intersection of CK and AB. Prove that $\triangle AKL$ is isosceles.

10. [Tournament of Towns 2007] Point B lies on a line with is tangent to circle ω at point A. The line segment $\overline{AB}$ is rotated about the center of the circle by some angle to form segment $\overline{A'B'}$. Prove that the line $\overline{AA'}$ bisects the segment $\overline{BB'}$.

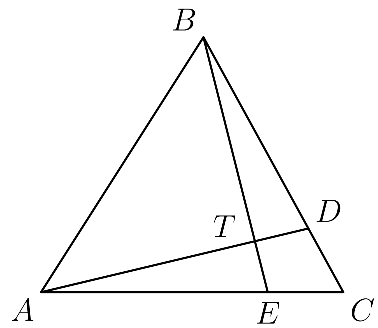
★ 11. [Romania 2006] Let ABC be a triangle. Points M, N on its sides AB, AC respectively, satisfy
$$\frac{BM}{AB} = 2 \cdot \frac{CN}{AC}.$$
The line perpendicular to MN passing through N intersects side BC at P. Prove that $\angle MPN = \angle NPC$.

★ 12. In isosceles $\triangle ABC$, $AB = AC$. Let K and L be points on $\overline{AB}$ and $\overline{AC}$ respectively such that $KL = BK + CL$. Let M be the midpoint of $\overline{KL}$. Draw a line through M parallel to $\overline{AC}$ and let its intersection with $\overline{BC}$ be N. Find, with proof, $\angle KNL$.

5 Area Ratios

1. Let G be the centroid of $\triangle ABC$. Show that the areas of $\triangle GAB$, $\triangle GBC$, and $\triangle GCA$ are equal. (Use the definition of centroid mentioned in the Tournament of Towns lecture problem.)

2. [AMC 10B 2004] In $\triangle ABC$ points D and E lie on $\overline{BC}$ and $\overline{AC}$, respectively. If $\overline{AD}$ and $\overline{BE}$ intersect at T so that $AT/DT = 3$ and $BT/ET = 4$, what is CD/BD?



3. Let ABC be a triangle. Points M and N lie on $\overline{AB}$ and AC respectively such that $AM = MB$ and $AN = 2AC$. Point P is the midpoint of $\overline{MN}$. Compute $[BPC] : [BAC]$.

   HINT: This is similar to a problem from the Menelaus section.

4. Let ABC be a triangle. Use Ceva's Theorem to show that the following sets of cevians are concurrent:

   - The medians from A, B, and C;
   - The angle bisectors from A, B, and C;
   - The lines AA_0, BB_0, CC_0, where A_0 is the tangency point of the incircle of $\triangle ABC$ with BC, and B_0 and C_0 are defined similarly.

5. In $\triangle ABC$, M, N, and P are placed on $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ such that the lines AM, BN, and CP are concurrent in a point S. Prove that

$$\frac{SM}{AM} + \frac{SN}{BN} + \frac{SP}{CP} = 1.$$

6. Points M and N on the sides $\overline{AB}$ and $\overline{AC}$ of $\triangle ABC$ satisfy $\overline{MN} \parallel \overline{BC}$. Prove that the lines BN and CM intersect on the A-median of $\triangle ABC$.

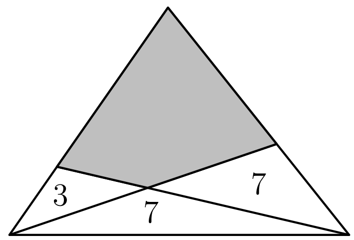
7. In $\triangle ABC$, let A_1 denote the foot of the altitude from A to BC, and suppose A_2 and A_3 are the midpoints of $\overline{BC}$ and $\overline{AA_1}$ respectively. Define B_1, B_2, B_3 and C_1, C_2, C_3 similarly. Show that $\overline{A_2A_3}$, $\overline{B_2B_3}$, and $\overline{C_2C_3}$ are concurrent.

   (Note: this concurrency point is more well known as the *symmedian point* of $\triangle ABC$.)

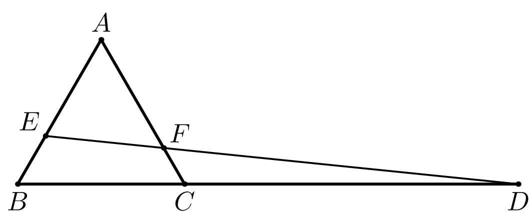
8. [CMIMC 2016] Let ABC be a triangle. The angle bisector of $\angle B$ intersects AC at a point P, while the angle bisector of $\angle C$ intersects AB at a point Q. Suppose the area of $\triangle ABP$ is 27, the area of $\triangle ACQ$ is 32, and the area of $\triangle ABC$ is 72. What is BC?

9. Let X be a point inside $\triangle ABC$ such that $[ABX] = [ACX]$. Prove that X lies on the A-median of the triangle.

10. [AMC 10B 2006] A triangle is partitioned into three triangles and a quadrilateral by drawing two lines from vertices to their opposite sides. The areas of the three triangles are 3, 7, and 7, as shown. What is the area of the shaded quadrilateral?



★ 11. [Purple Comet 2014] The diagram below shows equilateral $\triangle ABC$ with side length 2. Point D lies on ray $\overrightarrow{BC}$ so that $CD = 4$. Points E and F lie on $\overline{AB}$ and $\overline{AC}$, respectively, so that E, F, and D are collinear, and the area of $\triangle AEF$ is half of the area of $\triangle ABC$. Find $\frac{AE}{AF}$.



★ 12. [AIME 1992] In triangle ABC, A', B', and C' are on the sides BC, AC, and AB, respectively. Given that AA', BB', and CC' are concurrent at the point O, and that

$$\frac{AO}{OA'} + \frac{BO}{OB'} + \frac{CO}{OC'} = 92,$$

find

$$\frac{AO}{OA'} \cdot \frac{BO}{OB'} \cdot \frac{CO}{OC'}.$$

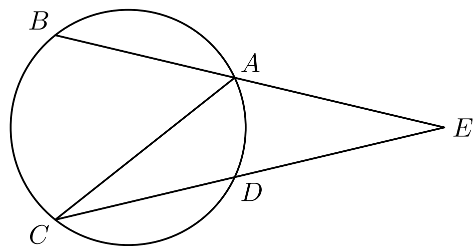
★ 13. [Equation of a Line in Barycentric Coordinates] Let ABC be a triangle, and suppose ℓ is a line passing through its interior. Show that there exist real numbers u, v, and w, not all equaling zero, such that

$$u[BPC] + v[CPA] + w[APB] = 0$$

for all points $P \in \ell$ inside $\triangle ABC$.

6 Circles

1. [AHSME 1977] In the figure, $\angle E = 40^\circ$, and $\widehat{AB}$, $\widehat{BC}$, and $\widehat{CD}$ have the same length. What is $\angle ACD$?



2. Complete the proof of the inscribed angle theorem in the case that O does not lie inside $\angle ACB$.

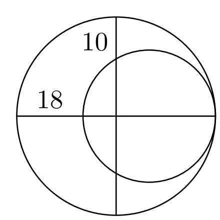
3. Let P be a point outside of a circle ω. Point A is placed on ω such that PA is tangent to ω, and a second line ℓ through P intersects ω at two distinct points X and Y with $PX < PY$. If $PX = 4$ and $XY = 5$, compute PA.

4. [AMC 10B 2008] Points A and B are on a circle of radius 5 and $AB = 6$. Point C is the midpoint of the minor arc AB. What is the length of the line segment AC?

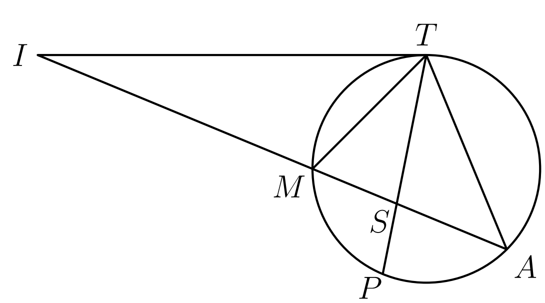
5. Let W, X, Y, and Z be four points on a circle in this order with $WZ = XY$, and suppose WZ intersects XY at a point P. If the measure of arc $\widehat{WXY}$ is 200°, compute the measure of $\angle P$ (also in degrees).

6. [HMMT 2006] Let A, B, C, and D be points on a circle such that $AB = 11$ and $CD = 19$. Point P is on segment $\overline{AB}$ with $AP = 6$, and Q is on segment $\overline{CD}$ with $CQ = 7$. The line through P and Q intersects the circle at X and Y. If $PQ = 27$, find XY.

7. [Math League HS 2002-2003] Two perpendicular diameters are drawn in a circle. Another circle, tangent to the first at an endpoint of one of its diameters, cuts off segments of lengths 10 and 18 from the diameters, as in the diagram (which is not drawn to scale). How long is a diameter of the larger circle?



8. In the figure below, TP bisects $\angle ATM$, and TI is tangent to $\odot(ATM)$ at T. Show that $SI = IT$.

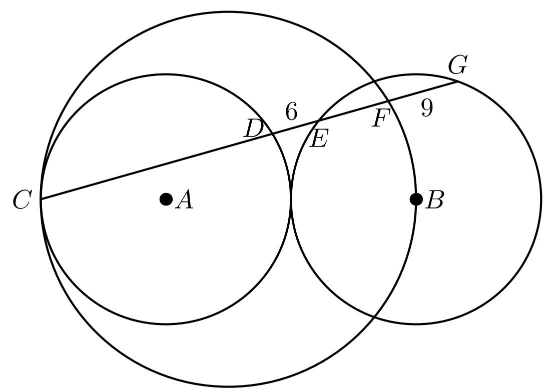


9. Inscribe equilateral triangle ABC inside a circle. Pick a point P on arc BC, and let AP intersect BC at Q. Prove that
$$\frac{1}{PQ} = \frac{1}{PB} + \frac{1}{PC}.$$

★ 10. [CMIMC 2017] Points A, B, and C lie on a circle Ω such that A and C are diametrically opposite each other. A line ℓ tangent to the incircle of $\triangle ABC$ at T intersects Ω at points X and Y. Suppose that $AB = 30$, $BC = 40$, and $XY = 48$. Compute $TX \cdot TY$.

(**Note:** Since we didn't cover how to do this yet, I will say that the radius of the incircle of $\triangle ABC$ is 10. Try to see how one might figure this out!)

★ 11. [Math Prize for Girls 2015] In the diagram below, the circle with center A is congruent to and tangent to the circle with center B. A third circle is tangent to the circle with center A at point C and passes through point B. Points C, A, and B are collinear. The line segment $\overline{CDEFG}$ intersects the circles at the indicated points. Suppose that $DE = 6$ and $FG = 9$. Find AG.



★ 12. [HMMT 2009] Points A and B lie on circle ω. Point P lies on the extension of segment AB past B. Line ℓ passes through P and is tangent to ω. The tangents to ω at points A and B intersect ℓ at points D and C respectively. Given that $AB = 7$, $BC = 2$, and $AD = 3$, compute BP.

7 Important Triangle Centers

1. Let ABC be a triangle with $\angle A = 50^\circ$, $\angle B = 60^\circ$, and $\angle C = 70^\circ$. Working with usual triangle center notation, find

   (a) $\angle AIB$, $\angle BIC$, and $\angle CIA$;

   (b) $\angle AHB$, $\angle BHC$, and $\angle CHA$;

   (c) $\angle AOB$, $\angle BOC$, and $\angle COA$.

2. Let ABC be a triangle with $\angle A = 120^\circ$, $\angle B = 20^\circ$, and $\angle C = 40^\circ$. Working with usual triangle center notation, find

   (a) $\angle AIB$, $\angle BIC$, and $\angle CIA$;

   (b) $\angle AHB$, $\angle BHC$, and $\angle CHA$;

   (c) $\angle AOB$, $\angle BOC$, and $\angle COA$.

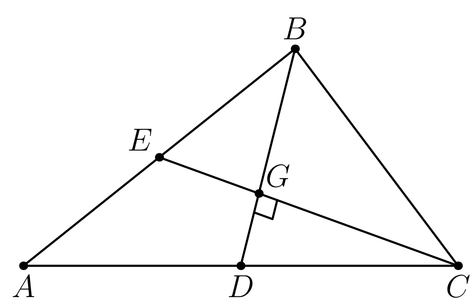
3. [Mandelbrot] Suppose $ABCD$ is a convex quadrilateral such that $AB = AC = BC = AD = 10$ and $BD = 13$. Compute the measure of $\angle BDC$.

   (**HINT:** Remember the important subdiagrams.)

4. In acute triangle $\triangle ABC$, let H denote its orthocenter and O its circumcenter. Show that $\angle BAH = \angle CAO$. Deduce that

$$\angle HAO = |\angle B - \angle C|.$$

5. [AHSME 1998] Medians BD and CE of triangle ABC are perpendicular, $BD = 8$, and $CE = 12$. What is the area of triangle ABC?



6. [HMMT 2015] Let ABC be a triangle with orthocenter H; suppose $AB = 13$, $BC = 14$, $CA = 15$. Let G_A be the centroid of triangle HBC, and define G_B, G_C similarly. Determine the area of triangle $G_AG_BG_C$.

   HINT: Let M_A, M_B, M_C be the midpoints of BC, CA, AB respectively. What can you say about $\triangle M_AM_BM_C$ in comparison to $\triangle G_AG_BG_C$?

7. In $\triangle ABC$, assume that the altitude from A and the median from A divide $\angle BAC$ into three equal parts. Find the angles of the triangle.

8. Let ABC be a triangle, and suppose D is a point on $\overline{BC}$. Let O_1 and O_2 be the circumcenters of $\triangle ABD$ and $\triangle ACD$ respectively. Show that

$$\triangle ABC \sim \triangle AO_1O_2.$$

9. [Part of AIME 2009] Let ABC be a triangle. Let D be a point in the interior of $\overline{BC}$. Let I_B and I_C denote the incenters of triangles ABD and ACD, respectively. The circumcircles of triangles BI_BD and CI_CD meet at distinct points P and D. Show that $\angle BPC$ is constant.

★ 10. [AMSP Team Contest 2015] Let $ABCD$ be an isosceles trapezoid with $AD = BC$, $AB = 6$, and $CD = 10$. Suppose the distance from A to the centroid of $\triangle BCD$ is 8. Compute the area of $ABCD$.

★ 11. [AIME 2011] Point P lies on the diagonal AC of square $ABCD$ with $AP > CP$. Let O_1 and O_2 be the circumcenters of triangles ABP and CDP respectively. Given that $AB = 12$ and $\angle O_1PO_2 = 120°$, compute AP.

★ 12. [AIME 2002] In triangle ABC the medians $\overline{AD}$ and $\overline{CE}$ have lengths 18 and 27, respectively, and $AB = 24$. Extend $\overline{CE}$ to intersect the circumcircle of ABC at F. Find the area of triangle AFB.

8 Triangle Computation, Part I

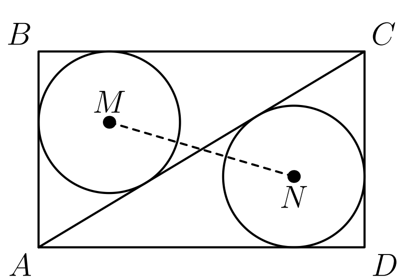
1. Let $\triangle ABC$ satisfy $AB = 13$, $BC = 14$, $CA = 15$. Compute

   - its area,
   - its inradius,
   - its circumradius,
   - its three exradii,
   - the values of $\cos A$, $\cos B$, and $\cos C$.

2. In $\triangle ABC$, $AB = 5$, $BC = 6$, and $\sin\angle BAC = \frac{3}{5}$. Compute $\sin\angle BCA$.

3. In $\triangle ABC$, $\angle BAC = 60°$, $BC = 7$, and $AB + AC = 13$. What is the area of the triangle?

4. [djmathman] Let $ABCD$ be a rectangle such that $AB = 3$ and $BC = 4$. Suppose that M and N are the centers of the circles inscribed inside triangles $\triangle ABC$ and $\triangle ADC$ respectively. What is MN^2?



5. Let ABC be a triangle and D a point on the side $\overline{BC}$. Show that

$$AB^2 - AC^2 = DB^2 - DC^2$$

if and only if $AD \perp BC$.

6. [AMC 12A 2003] An object moves 8 cm in a straight line from A to B, turns at an angle α, measured in radians and chosen at random from the interval $(0, \pi)$, and moves 5 cm in a straight line to C. What is the probability that $AC < 7$?

7. [NIMO 11] In triangle ABC, $\sin A \sin B \sin C = \frac{1}{1000}$ and $AB \cdot BC \cdot CA = 1000$. What is the area of triangle ABC?

   (**HINT:** How does $AB \cdot BC \cdot CA$ relate to the area of $\triangle ABC$?)

8. [djmathman] Two chords are constructed in a circle with radius 4 such that their intersection lies on the circle, and the angle formed by the two chords has measure $30°$. One of the chords has length 5. What is the largest possible value for the length of the other chord?

9. [AIME 2003] Find the area of rhombus $ABCD$ given that the radii of the circles circumscribed around triangles ABD and ACD are 12.5 and 25, respectively.

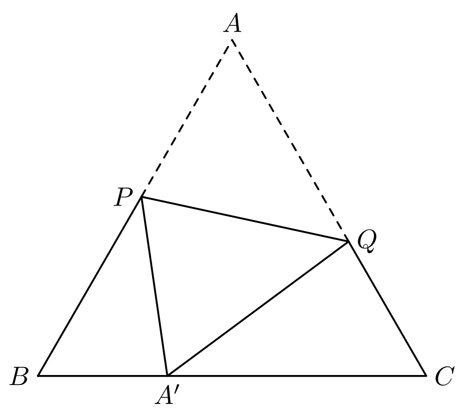
10. Let ABC be an isosceles triangle with $AB = AC$, and suppose D is a point which lies on $\overline{BC}$. Must the circumradii of $\triangle ABD$ and $\triangle ACD$ be equal to each other?

11. Let $\mathcal{T}$ be a triangle with inradius r and exradii r_a, r_b, and r_c. Show that

$$[\mathcal{T}] = \sqrt{r r_a r_b r_c}.$$

12. [Mandelbrot] In triangle ABC, $AB = 5$, $AC = 6$, and $BC = 7$. If point X is chosen on BC so that the sum of the areas of the circumcircles of triangles AXB and AXC is minimized, then determine BX.

★ 13. [AHSME 1991] Equilateral triangle ABC has been creased and folded so that vertex A now rests at A' on $\overline{BC}$ as shown. If $BA' = 1$ and $A'C = 2$ then what is the length of crease $\overline{PQ}$?



★ 14. [HMMT 2004] Right triangle XYZ has right angle at Y and $XY = 228$, $YZ = 2004$. Angle Y is trisected, and the angle trisectors intersect XZ at P and Q so that X, P, Q, and Z lie on XZ in that order. Find the value of $(PY + YZ)(QY + XY)$.

★ 15. [AMC 12B 2013] Let ABC be a triangle where M is the midpoint of $\overline{AC}$, and $\overline{CN}$ is the angle bisector of $\angle ACB$ with N on $\overline{AB}$. Let X be the intersection of the median $\overline{BM}$ and the bisector $\overline{CN}$. In addition $\triangle BXN$ is equilateral and $AC = 2$. What is BN^2?

9 Triangle Computation, Part II: Trigonometric Edition

1. Using some of the formulas from class, compute $\cos 75°$ and $\sin 75°$. Check that
$$\cos^2 75° + \sin^2 75° = 1.$$

2. Using both a subtraction angle identity and a half angle identity, compute the value of $\tan 15°$ in two different ways.

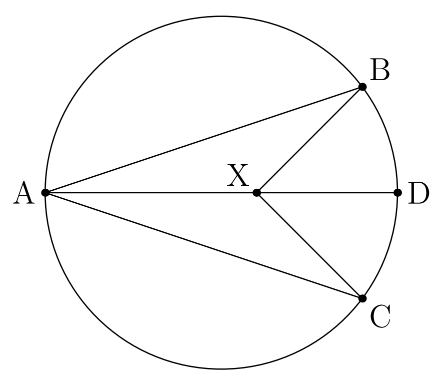
3. [AMC 12A 2012] A triangle has area 30, one side of length 10, and the median to that side of length 9. Let θ be the acute angle formed by that side and the median. What is $\sin\theta$?

4. For all angles θ, show that
$$\sin(3\theta) = 3\sin\theta - 4\sin^3\theta$$
by using the sine angle sum identity multiple times.

5. [AIME 2013] In equilateral $\triangle ABC$ let points D and E trisect $\overline{BC}$. Compute $\sin(\angle DAE)$.

6. [AMC 12A 2014] Two circles intersect at points A and B. The minor arcs AB measure $30°$ on one circle and $60°$ on the other circle. What is the ratio of the area of the larger circle to the area of the smaller circle?

7. [AHSME 1993] Points A, B, C and D are on a circle of diameter 1, and X is on diameter $\overline{AD}$. If $BX = CX$ and $3\angle BAC = \angle BXC = 36°$, show that
$$AX = \cos 6° \sin 12° \sec 18°.$$



8. [NIMO 12] Triangle ABC has sidelengths $AB = 14, BC = 15$, and $CA = 13$. We draw a circle with diameter AB such that it passes BC again at D and passes CA again at E. Find the circumradius of $\triangle CDE$.

9. [AIME 2004] A circle of radius 1 is randomly placed in a 15-by-36 rectangle $ABCD$ so that the circle lies completely within the rectangle. Compute the probability that the circle will not touch diagonal AC.

10. Prove the *Law of Tangents*: in any $\triangle ABC$, we have

$$\frac{\tan\frac{A+B}{2}}{\tan\frac{A-B}{2}} = \frac{a+b}{a-b}.$$

11. Let ABC be a triangle, and suppose D, E, and F are points on $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ respectively. Show that AD, BE, and CF are concurrent if and only if

$$\frac{\sin\angle CAD}{\sin\angle BAD}\cdot\frac{\sin\angle ABE}{\sin\angle CBE}\cdot\frac{\sin\angle BCF}{\sin\angle ACF} = 1.$$

Note that this gives us another criterion for showing three cevians are concurrent; this one is more useful when we have information about the angles as opposed to information about the sides.

★ 12. [Mandelbrot] Triangle ABC has sides of length $AB = 6$, $AC = 5$, and $BC = 4$. Compute

$$\frac{\cos\frac{1}{2}(A-B)}{\sin\frac{1}{2}C} - \frac{\sin\frac{1}{2}(A-B)}{\cos\frac{1}{2}C}.$$

★ 13. [ARML 1983] In an isosceles triangle, the altitudes intersect on the inscribed circle. Compute the cosine of the vertex angle.

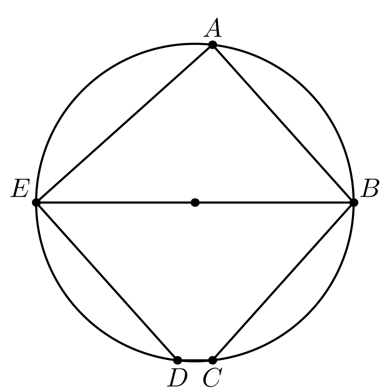
★ 14. Show that in $\triangle ABC$, we have

$$BC^3\cos(B-C) + CA^3\cos(C-A) + AB^3\cos(A-B) = 3(BC)(CA)(AB).$$

10 Cyclic Quadrilaterals

1. Suppose ABC is a right triangle with a right angle at B. Point D lies on side $\overline{AB}$, and E is the foot of the perpendicular from D to AC. If $\angle BAC = 17^\circ$ and $\angle ABE = 23^\circ$, compute $\angle DCB$.

2. [AMC 10B 2011] In the given circle, the diameter $\overline{EB}$ is parallel to $\overline{DC}$, and $\overline{AB}$ is parallel to $\overline{ED}$. The angles AEB and ABE are in the ratio $4:5$. What is the degree measure of angle BCD?



3. Suppose that P is a point on minor arc $\widehat{BC}$ of the circumcircle of an equilateral triangle ABC. Show that
$$PA = PB + PC.$$

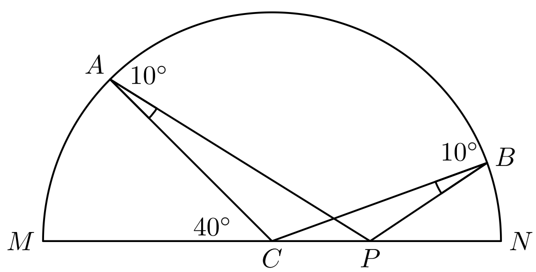
4. Let ABC be a right triangle with $\angle B = 90°$. Square is $ACDE$ is erected outside $\triangle ABC$. Let O be the center of this square. Find $\angle OBC$.

5. Let $\triangle ABC$ be a scalene triangle. Points D, E, F are the feet of the altitudes from A, B, C to BC, CA, AB respectively. Prove that the orthocenter of $\triangle ABC$ is the incenter of $\triangle DEF$.

6. [Miquel's Theorem] Let ABC be a triangle, and let D,E, and F be points on line segments $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ respectively. Prove that the circumcircles of triangles AEF, BFD, and CDE are concurrent (i.e. meet at a common point).

   (**Hint:** Let P be the point inside $\triangle ABC$ at which two of the circles intersect and show that the third circumcircle passes through it as well.)

7. [AHSME 1983] Distinct points A and B are on a semicircle with diameter MN and center C. The point P is on CN and $\angle CAP = \angle CBP = 10°$. If $\widehat{MA} = 40°$, then what is $\widehat{BN}$?



8. [Ray Li] In triangle ABC, $AB = 36$, $BC = 40$, $CA = 44$. The bisector of angle A meets BC at D and the circumcircle at E different from A. Calculate the value of DE^2.

9. [Bulgaria 1993] A parallelogram $ABCD$ with an acute angle BAD is given. The bisector of $\angle BAD$ intersects CD at point L and the line BC at point K. Prove that the circumcenter of $\triangle LCK$ lies on the circumcircle of $\triangle BCD$.

10. [AMC 10/12B 2013] In triangle ABC, $AB = 13$, $BC = 14$, and $CA = 15$. Distinct points D, E, and F lie on segments $\overline{BC}$, $\overline{CA}$, and $\overline{DE}$, respectively, such that $\overline{AD} \perp \overline{BC}$, $\overline{DE} \perp \overline{AC}$, and $\overline{AF} \perp \overline{BF}$. The length of segment $\overline{DF}$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

11. [CMIMC 2017] Cyclic quadrilateral $ABCD$ satisfies $\angle ABD = 70°$, $\angle ADB = 50°$, and $BC = CD$. Suppose AB intersects CD at point P, while AD intersects BC at point Q. Compute $\angle APQ - \angle AQP$.

12. [APMO 2007] Let ABC be an acute angled triangle with $\angle BAC = 60°$ and $AB > AC$. Let I be the incenter and H the orthocenter of the triangle ABC. Prove that $2\angle AHI = 3\angle ABC$.

13. [Sharygin 2008] Quadrilateral $ABCD$ is circumscribed about a circle with center I. Prove that the projections of points B and D to the lines IA and IC lie on a single circle.

★ 14. [AIME 2016] Circles ω_1 and ω_2 intersect at points X and Y. Line ℓ is tangent to ω_1 and ω_2 at A and B, respectively, with line AB closer to point X than to Y. Circle ω passes through A and B intersecting ω_1 again at $D \neq A$ and intersecting ω_2 again at $C \neq B$. The three points C, Y, D are collinear, $XC = 67$, $XY = 47$, and $XD = 37$. Find AB^2.

★ 15. [Mandelbrot 2011-2012] Let ABC be a triangle inscribed in circle ω with $BC = 17$. The angle bisector of $\angle BAC$ intersects ω again at P. Given that $\sin \angle ABP = \frac{3}{5}$, and that the radius of ω is 20, compute the area of quadrilateral $ABPC$.

11 Frequent Configurations

Instead of putting stars all over the place, we warn you in advance that most of the problems in this section are quite challenging.

1. As a warm up, a few angle chasing questions regarding midpoints of the arcs. Let M_A denote the midpoint of the arc $\widehat{BC}$ not containing A, and define M_B and M_C similarly.

   (a) Compute $\angle M_ABC$, $\angle M_ACB$, $\angle M_AM_BM_C$, and $\angle ACM_B$ in terms of the measures of the angles of $\triangle ABC$.

   (b) Show that $AM_A \perp M_BM_C$. (**HINT:** these are chords!)

   (c) Use the previous part to deduce that I is the orthocenter of $\triangle M_AM_BM_C$.

2. [CGMO 2012] Let ABC be an acute triangle. The incircle of ABC is tangent to sides AB and AC at D and E respectively, and O is the circumcenter of BCI. Prove that $\angle ODB = \angle OEC$.

3. Let ABC be an acute triangle. Show that $AH = AO$ iff $\angle BAC = 60°$. (Do this using two methods: $AH = 2R \cos A$ and Fact 5)

4. [NIMO 4] In cyclic quadrilateral $ABXC$, $\measuredangle XAB = \measuredangle XAC$. Denote by I the incenter of $\triangle ABC$ and by D the projection of I on $\overline{BC}$. If $AI = 25$, $ID = 7$, and $BC = 14$, then compute XI.

5. Let ABC be a triangle with orthocenter H. Must the circumradii of $\triangle BAC$ and $\triangle BHC$ be equal?

6. [AIME 2016] In $\triangle ABC$ let I be the center of the inscribed circle, and let the bisector of $\angle ACB$ intersect AB at L. The line through C and L intersects the circumscribed circle of $\triangle ABC$ at the two points C and D. If $LI = 2$ and $LD = 3$, then $IC = \frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

7. [CMIMC 2016] In parallelogram $ABCD$, angles B and D are acute while angles A and C are obtuse. The perpendicular from C to AB and the perpendicular from A to BC intersect at a point P inside the parallelogram. If $PB = 700$ and $PD = 821$, what is AC?

8. [NIMO Summer Contest 2015] Let $\triangle ABC$ be a triangle with $AB = 85$, $BC = 125$, $CA = 140$, and incircle ω. Let D, E, F be the points of tangency of ω with $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ respectively, and furthermore denote by X, Y, and Z the incenters of $\triangle AEF$, $\triangle BFD$, and $\triangle CDE$, also respectively. Find the circumradius of $\triangle XYZ$.

9. [Sharygin Final Round 2012] In acute triangle ABC inscribed in circle ω, let A' be the projection of A onto BC and B', C' the projections of A' onto AC, AB respectively. Line $B'C'$ intersects ω at X and Y and line AA' intersects ω for the second time at D. Prove that A' is the incenter of triangle XYD.

★ 10. [China Western MO 2012] Let O be the circumcircle of acute $\triangle ABC$, H be the orthocentre. Let AD be the altitude of $\triangle ABC$ from A, and let the perpendicular bisector of AO intersect BC at E. Prove that the circumcircle of $\triangle ADE$ passes through the midpoint of OH.

★ 11. [Mandelbrot 2007-2008] Altitudes $\overline{AP}$ and $\overline{BQ}$ of an acute triangle $\triangle ABC$ intersect at point H. If $HP = 5$ while $HQ = 2$, then calculate

$$(BP)(PC) - (AQ)(QC).$$

★ 12. Suppose the Euler line of an acute triangle $\triangle ABC$ is parallel to BC. Show that

$$(\tan B)(\tan C) = 3.$$