Mass Points and Stewart's Theorem: Facts

Math Circle Competition Team

September 25th, 2016

- **Vocabulary.**

  Cevian - Any line segment in a triangle with one endpoint on a vertex of the triangle and the other endpoint on the opposite side. Medians, altitudes, and angle bisectors are special cases of cevians.

  Median - A line segment joining a vertex of a triangle to the midpoint of the opposing side.

  Altitude - A line segment passing through one vertex of a triangle and perpendicular to the opposite side.

  Angle Bisector - A line or ray that divides an angle into two congruent angles.

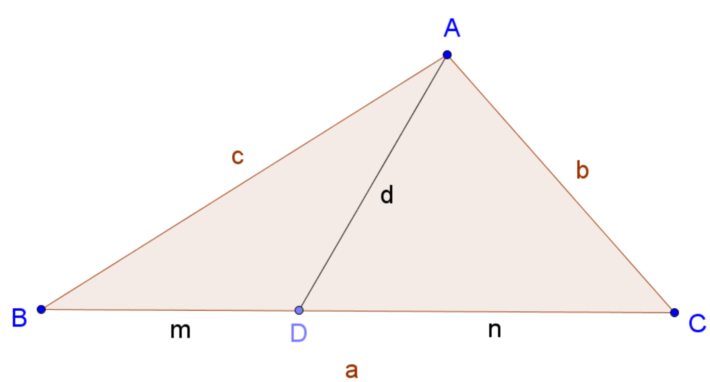
- **Stewart's Theorem (General).** Let $\triangle ABC$ be a triangle with sides of length a, b, c which are opposite vertices A, B, C, respectively. If segment AD is drawn so that D is on BC, $BD = m$, $DC = n$ and $AD = d$, we have that

$$b^2m + c^2n = amn + d^2a,$$

  which is sometimes written as

$$man + dad = bmb + cnc.$$

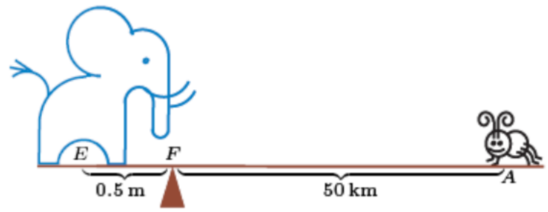
  In this form, Stewart's formula can be remembered as "A *man* and his *dad* found a *bomb* in the *sink*." A properly labeled triangle is shown in the diagram below:



- **Apollonius' Theorem.** In a special case of Stewart's Theorem where AD is a median (so $BD = DC = m$), we have the simplified formula

$$b^2 + c^2 = 2(m^2 + d^2).$$

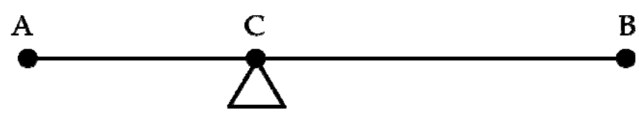
- **Mass Points.** Mass point geometry is a physics tool that lets us find ratios of distances in triangles. Let $m(E)$ denote the mass at a point E, and consider the diagram below:



A seesaw balances only when the product of the mass and the distance to the fulcrum on both sides is equal. In equation form, this is represented as

$$m(E) \cdot EF = m(A) \cdot FA.$$

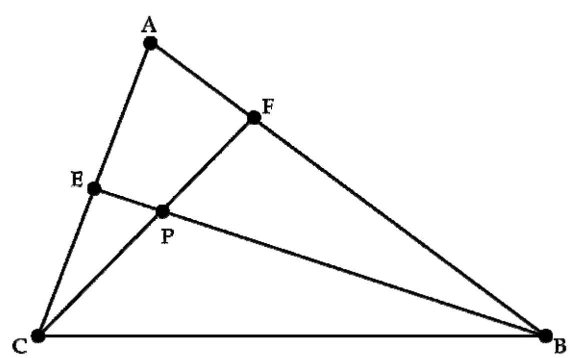
We can apply this to the sides of a triangle if we assume that points on the triangle have masses. If AB is a side of the triangle and C is a point on AB, we have:



where AB will balance at C if $m(A) \cdot AC = m(B) \cdot BC$. We also have the following *addition relation*:

$$m(A) + m(B) = m(C),$$

where C is the center of mass of A and B. As an example, suppose that $AE : EC = 1 : 1$ and $AF : FB = 1 : 3$ in the diagram below. We wish to find $EP : PB$.



Since we want AC to balance at E, we have that $m(A) \cdot AE = m(C) \cdot EC$, or $m(A) = m(C)$, so let $m(A) = m(C) = 1$. Then by the addition relation, we have $m(E) = m(A) + m(C) = 2$. How can we find the mass at B by using F? And what does this tell us about $EP : PB$?