Expected Area of Inscribed Polygons

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MATH CIRCLE

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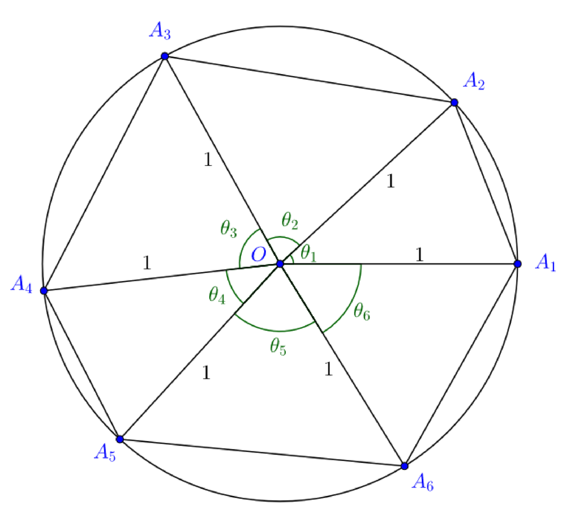
Introduction

Our objective is to find the expected area of a convex polygon formed by n randomly chosen points on a unit circle. We will utilize calculus and probability distribution functions to solve the problem and find an equation in terms of n that gives the expected value.

Finding Area

We first define an equation that gives the area of the n-gon. As an example, we place 6 random points $(A_1, A_2, A_3, A_4, A_5, A_6)$ on a unit circle and connect them to the center O of the unit circle. The area of the polygon is just the sum of the areas of the triangles.



Note that the area of a triangle is $\frac{1}{2}ab \cdot \sin\theta$, where a and b are two sides of a triangles and θ is the angle between them. Without loss of generality we let the first point, A_1, to be at (1,0). Now, we can define the location of each point A_x as its reference angle, θ_x from A_{x-1}. As a = b = 1, we find the area of the triangles to be:

$\frac{1}{2}(\sin(\theta_1) + \sin(\theta_2) + \sin(\theta_3) + \sin(\theta_4) + \sin(\theta_5) + \sin(\theta_6))$.

What if instead of 6 points on the unit circle there are n points? We can use the same process to generalize the area:

$$\begin{aligned}&[A_1A_2A_3 \dots A_{n-1}A_n] \\ &= [A_1OA_2] + [A_2OA_3] + \cdots + [A_{n-1}OA_n] + [A_nOA_0] \\ &= \frac{1}{2}\left(\sum_{i=1}^{n} \sin(\theta_i)\right)\end{aligned}$$

Expected Value

The expected area of $A_1A_2A_3 \dots A_{n-1}A_n$ is

$$\mathbb{E}([A_1A_2A_3 \dots A_{n-1}A_n]) = \mathbb{E}\left(\sum_{i=1}^{n} \sin(\theta_i)\right)$$

where $\mathbb{E}(p)$ is the expected value of a random variable p. Since the $n-1$ points (not including A_1) are placed on the circle with uniform distribution, the expected value of the angle between any two adjacent points would be the same. That implies

$\mathbb{E}(\sin\theta_1) = \mathbb{E}(\sin(\theta_k))$ for all k. Therefore the expected value simplifies to $\mathbb{E}\left(\frac{1}{2}(n\sin\theta_1)\right)$.

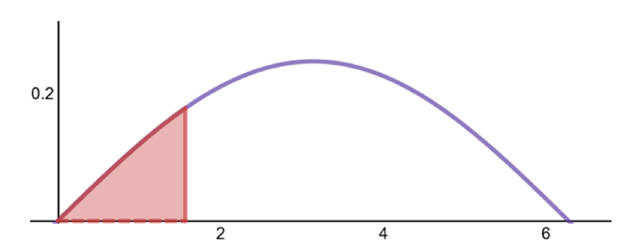
Probability Distribution of θ_1

The expected value $\mathbb{E}\left(\frac{n}{2}\sin\theta_1\right)$ is equivalent to $\frac{n}{2}\mathbb{E}(\sin\theta_1)$ because the continuous random variable θ_1 is independent of n. In order to find the expected value we add up the probabilities of each θ_1 occurring times $\frac{n}{2}\sin\theta_1$. Because there are infinitely many possible values of θ_1 we have to integrate to sum all the probabilities. So,

$$\mathbb{E}(\sin\theta_1) = \int_0^{2\pi} f(\theta_1) \cdot \sin\theta_1 \, d\theta_1$$

where $f(\theta_1)$ is the probability that the minimum angle formed is θ_1. The function $f(\theta_1)$ is also referred to as the probability density function.

Let's say that our probability density function looks like this:



where the x-axis is the value of θ_1 and the y-axis is the probability of it occurring.

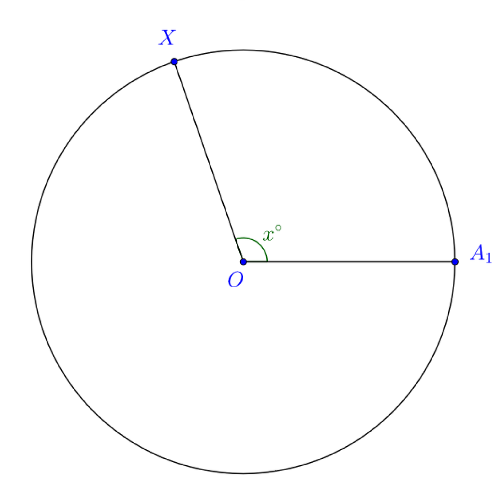
If we wish to find $P\left(0 \le \theta_1 \le \frac{\pi}{2}\right)$ we just need to find the area under the density function from 0 to $\frac{\pi}{2}$ (The part shaded in red).

Generally, $P(0 \le \theta_1 \le x)$ would just be the area under $f(\theta_1)$ from 0 to x or $\int_0^x f(\theta_1)\, d\theta_1$. We can call this function $F(x)$, also known as the probability distribution function. So if we find the probability distribution function we can find its derivative to arrive at the probability density function.

Finding the Probability Distribution Function

The probability distribution function, $F(x)$, is

$P(0 \le \theta_1 \le x) = \int_0^x f(\theta_1)\, d\theta_1$. In words, $F(x)$ is the probability that θ_1 is less than or equal to x.



Let the point X be on the unit circle such that $\angle A_1OX = x$. For θ_1 to be less than or equal to x, at least one of the $n-1$ randomly chosen points must be inside of arc A_1X. So the probability that at least one of the points is inside arc A_1X would be $1 -$ [the probability that all the points are outside the arc]. Since the angle of the arc is x, the probability that a randomly chosen point on the circle is outside the arc is $\frac{2\pi - x}{2\pi}$. Thus the probability that all $n-1$ points are outside the arc is $\left(\frac{2\pi-x}{2\pi}\right)^{n-1}$.

Finally, if $0 \le x \le 2\pi$ the probability that $\theta_1 \le x$ is $1 - \left(\frac{2\pi-x}{2\pi}\right)^{n-1}$. Note that since $0 \le \theta_1 \le 2\pi$, then if $x > 2\pi$ then the probability would be 1 and if $x < 0$ the probability would be 0.

So now we can define our probability distribution function to be:

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - \left(\dfrac{2\pi - x}{2\pi}\right)^{n-1} & 0 \le x \le 2\pi \\ 1 & x > 2\pi \end{cases}$$

Now that we have found the distribution function we need to find its derivative to arrive at the density function $f(\theta_1)$. Applying this yields the function to be:

$$f(\theta_1) = \begin{cases} 0 & \theta_1 < 0 \\ \left(\frac{n-1}{2\pi}\right)\left(\frac{2\pi-\theta_1}{2\pi}\right)^{n-2} & 0 \le \theta_1 \le 2\pi \\ 0 & \theta_1 > 2\pi \end{cases}$$

Finding the Expected Value Equation

Now that we have found the probability density function of θ_1, we multiply this by $\sin\theta_1$and integrate to get the expected value:

Thus,

$$\begin{aligned}\frac{n}{2}\mathbb{E}(\sin\theta_1) &= \frac{n}{2}\int_{-\infty}^{\infty} f(\theta_1)\sin\theta_1 \, d\theta_1 \\ &= \frac{n}{2}\int_0^{2\pi}\left(\frac{n-1}{2\pi}\right)\left(\frac{2\pi-\theta_1}{2\pi}\right)^{n-2} \sin\theta_1\, d\theta_1\end{aligned}$$

Because the probability density function's value is 0 when θ_1 is outside the range $[0,2\pi]$ we can remove that part out of the integral.

Removing constant terms and substituting $u = 2\pi - \theta_1$ simplifies to

$$\frac{n(n-1)}{2(2\pi)^{n-1}}\int_0^{2\pi}(2\pi-\theta_1)^{n-2}\sin\theta_1 \, d\theta_1 = \frac{-n(n-1)}{2(2\pi)^{n-1}}\int_0^{2\pi}(u)^{n-2}\sin u \, du$$

Applying the tabular method of integration by parts we represent this integral as a sum:

$$\mathbb{E}([A_1A_2A_3 \dots A_{n-1}A_n]) = \pi(n!)\sum_{i=1}^{\lfloor\frac{n-1}{2}\rfloor}\frac{(-1)^{i+1}}{(n-2i)!\,(2\pi)^{2i}}$$

Using our formula we compute the expected values for specific values of n:

n	$\mathbb{E}(\boldsymbol{Area})$	n	$\mathbb{E}(\boldsymbol{Area})$
3	0.47746	20	2.89123
4	0.95493	50	3.09546
5	1.3497	100	3.12959
10	2.3762	1000	3.14146
7,000,000	3.141592653587		

As you can see, as n approaches infinity, the expected value is approaching π, since the polygon will approximate the unit circle.

Further Questions

1. What is the expected volume of a $3D$ shape with n vertices made from random points chosen on a unit sphere? What about $4D$?
2. What is the expected perimeter of an n-gon made from choosing n random points on a unit circle?
3. What is the expected area of a polygon circumscribed about a unit circle?

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