Shhh! It's a Secret

Graph-Theoretic Analysis of a Handshake Problem Variant

Duaily Lopez-Reyes and Madeline Munson

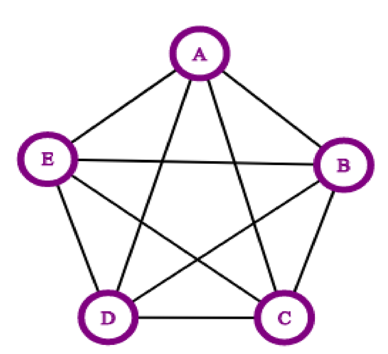
MATH CIRCLE

EMORY COLLEGE OF ARTS AND SCIENCES

Background

Many know about the infamous "Handshake Problem": in a room of n people, how many handshakes must happen so that everyone shakes everybody else's hands (not being able to shake hands with yourself or with someone else more than once)? The answer to this question is a simple one: the number of edges of the complete graph on n vertices (K_n).

For example, when $n = 5$:



$$|E| = \binom{n}{2} = \frac{n(n-1)}{2}$$

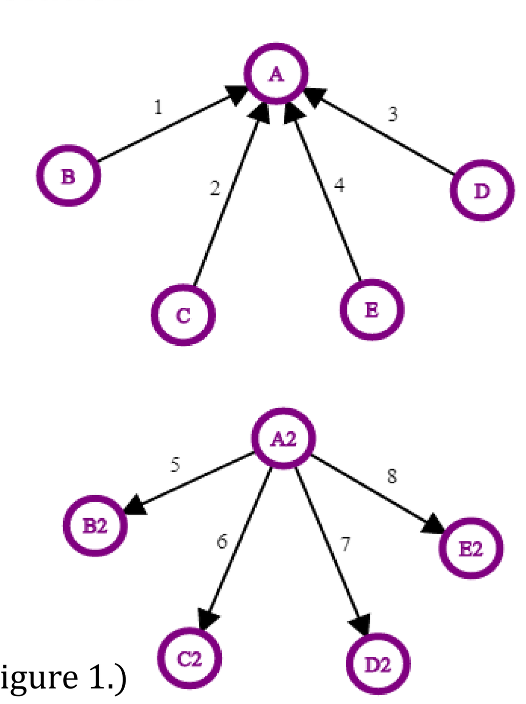
(A K_5 graph modeling the problem)

Our Variant

Each person in a group has one secret. Every time two people converse, they share all the secrets they know and gain the other person's secret(s). How many conversations must be held so that everyone knows everyone else's secrets? Can a general formula be created which takes an input of n, or amount of people (vertices), and outputs the optimal amount of conversations, $|E|$ (edges)? If so, what is that formula?

At First Glance

When first challenged with the problem, an obvious answer is to have one person that collects all of the secrets and then shares them back to everybody else. This would require all people (n) to have 2 conversations (|E|) with the "main vertex" (in Figure 1, vertex A). However, this is not the best solution.



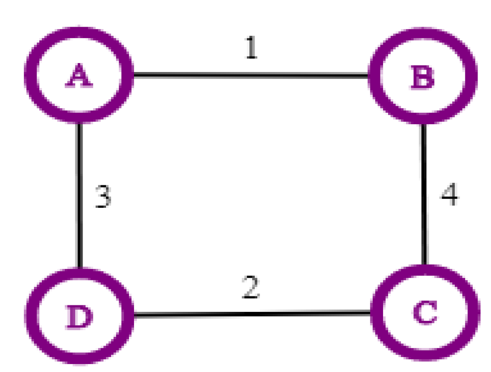
(Figure 1.)

n	|E|
1	0
2	2
3	4
4	6
5	8
6	10
7	12
8	14
9	16
10	18
n	*2n-2*

Optimal Strategy

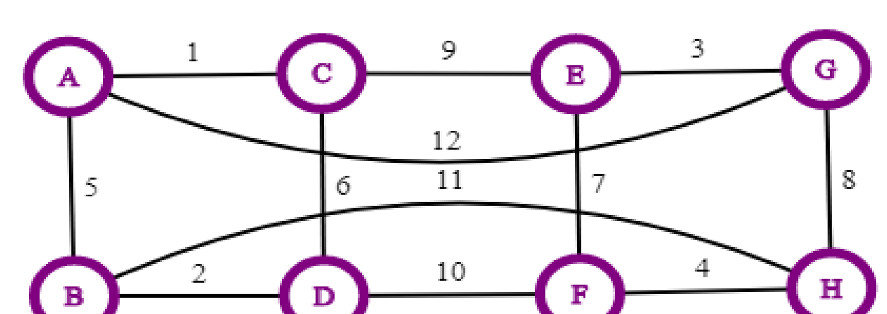
The essential part of producing the optimal amount of conversations is order. In the case of 4 people, there are 4 vertices, ie. A, B, C, and D. For optimal results, A must talk to B and C to D before A or B talk to C or D. After A and B have shared and C and D have shared, A may talk to D and B may talk to C, therefore making 4 conversations to share all secrets. (Figure 2)

(Figure 2.)

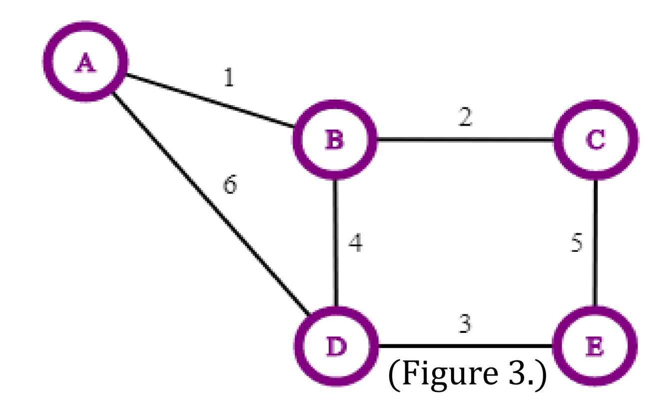


This proves that the most efficient way for four people (further referred to as vertices) to know everybody else's secrets is through 4 conversations (further referred to as edges). This will be the basis for the rest of the n-vertex cases.

For the 5-vertex case, we can split up the vertices into a group of four and a single outlier that starts the conversation, telling one member of the group of 4 their secret. This is followed by the 4-group completing the optimal cycle of secret sharing (Figure 2) allowing everyone in the group of four to know all secrets. Then, to finish the loop, someone must go back and tell the person who began the conversation so that person learns the rest of the secrets. (Figure 3)

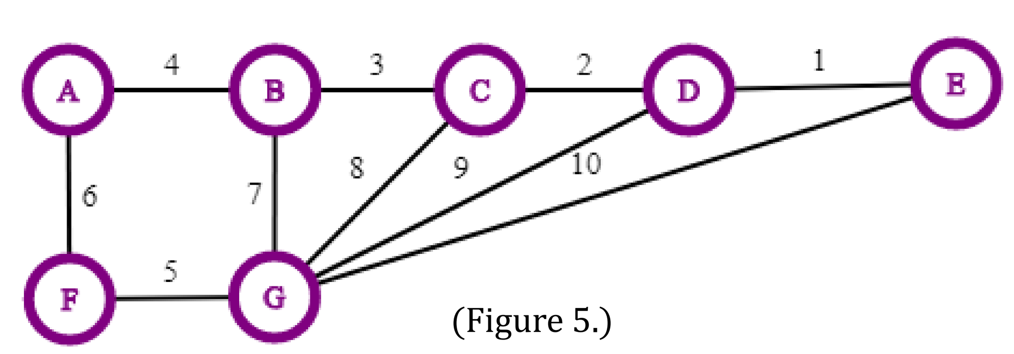


(Figure 4.)



(Figure 3.)

An interesting case is a group of 8 people. Here it is expected the number of conversations needed would change for each added edge because of the double groups of 4. Surprisingly it stays the same. (Figure 4)



(Figure 5.)

Through our process we originally considered making as many groups of 4 as possible, then treating the rest as outliers that start the conversations. We found that after 11 people, this strategy caused oversharing. The way to avoid oversharing and still achieve our $2n - 4$ outcome is by creating a group of 4 or 8 and putting the rest of the people out in a chain. This is where the $2n - 4$ comes from. (Figure 5)

n	|E|
1	0
2	1
3	3
4	4
5	6
6	8
7	10
8	12
9	14
10	16
n	*2n-4*

This table shows the relation between the amount of people and the number of optimal conversations. Our results for this table are significantly better than the results at first glance. This is the result of the optimal algorithm that we found splitting up the vertices into a group of 4 or 8 and chaining the outliers together leading into the conversation (each new vertex introduced adds 2 more conversations). This yielded $2n - 4$ for $n \geq 4$ as our solution.
We are yet to come up with a concrete proof for our conjecture as our problem starts to diverge from regular graph theory and more into time-dependent graph theory which is a field that is not very well studied.

Conclusion

Through our methods, we conjecture that $2n - 4$ is the least amount of conversations that are necessary for n people to share all of their secrets, if every time 2 people converse they have to share all of their previously known secrets. In other words,

$$2n - 4 \text{ for } n \in \mathbb{N}, n \geq 4$$

is the optimal amount of edges required for n vertices to satisfy our problem.

Further Research and Questions

- Since we do not have a proof for our conjecture, we cannot claim that it is the ultimate solution to the problem. This leads us to question our findings and believe that there may be an even better solution containing less conversations than currently.
- This problem does not follow the fundamental rules of graph theory in that these graphs are time-dependent. For example, some edges between vertices must happen before others so that the flow of information is not only complete but done in the least number of conversations. This makes graph theory the *best* representation of this problem, not the *exact* representation. This raises questions in a field of mathematics that is not well studied, **time-dependent graph theory.**
- When we conjectured $2n - 4$ for $n \geq 4$, we thought about a method of proving this so that there is no possibility for ambiguity. The first problem was the idea that these graphs are time-dependent, but even when factoring that big part out, we are not sure where to go with this proof. Mr. Sudip Sinha suggests studying an upper/lower bound for the solution. We are not aware of the correct path for that proof.
- Another difficult part about this problem being time-dependent is the need for notation for the flow of information and a counting system that evolves over time in order to count how many secrets each vertex knows.

Acknowledgements

We would like to thank Math Circle for allowing us the opportunity to do this project. Furthermore, we would like to thank Mr. Brad Elliott, Mr. Sudip Sinha, Mr. Bryson Kagy, Mr. Steven Olsen, and all the other instructors at Math Circle for the immense amount of help at each step of the process.