2019 Tri-Math Tournament

Problem Solutions

Math Circle Summer Program

1. *A seven-inch segment of ginger is cut*
   Into sizes of three, one, or two.
   The number of ways to accomplish this feat
   Is the same as the ounces to brew.
   The order of these is beside the point,
   2-2-3s the same as 3-2-2.

   $\boxed{8}$ The eight ways are: 3-3-1, 3-2-2, 3-2-1-1, 3-1-1-1-1, 2-2-2-1, 2-2-1-1-1, 2-1-1-1-1-1, 1-1-1-1-1-1-1.

2. *Distinct arrangements you must compile,*
   To find the ounces of the bile.
   Compute, for dillo this unique fact,
   The same for arma, then subtract.

   $\boxed{48}$ For "dillo", we have 5 choices for the first letter, 4 for the second, and so on. This is $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$, sometimes written as 5! or 5 factorial. However, since the letter l appears twice, we have counted some possibilities twice; since we can arrange the l's in 2! ways, we divide by 2! to eliminate these duplicates. Thus the number of distinct arrangements of the word "dillo" is $\frac{5!}{2!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1} = 5 \cdot 4 \cdot 3 = 60$.

   Similarly, for "arma", we have $4 \cdot 3 \cdot 2 \cdot 1 = 4!$ choices, but there are 2! ways to arrange the two a's, for a total of $\frac{4!}{2!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1} = 4 \cdot 3 = 12$ arrangements. Subtracting these yields $60 - 12 = \boxed{48}$.

3. *The beetles' number you'll soon see;*
   two are left over when grouped by three.
   Six bugs remain when grouped by sevens
   and six again grouped by elevens.
   Though many choices fit these guides,
   The least of these you should supply.

   $\boxed{83}$ Since 6 bugs are left over when grouped by 11's, our possibilities are $17, 28, 39, 50, 61, 72, 83, 94, \ldots$ The first of these which also leaves a remainder of 6 when divided by 7 is 83, and 83 also leaves a remainder of 2 when divided by 3, so the answer is $\boxed{83}$.

4. As part of their new enterprise, the twins want to create a line of fireworks known as Weasleys Wildfire Whiz-bangs. In one of their designs, the firework is launched into the sky, where it creates some number of straight lines of light in a single plane parallel to the ground. The twins want these lines to intersect exactly 66 times. If no two lines are parallel and no three lines intersect at the same point, how many lines should the firework create to achieve this number of intersections?

   $\boxed{12}$ Since no two lines are parallel and no three lines intersect at the same point, every pair of lines intersects exactly once. Thus the number of intersections is the same as the number of unique pairs of lines. If the number of intersection is n, this is "n choose 2", or $\binom{n}{2}$. This gives $\binom{n}{2} = 66$, which we can solve using the formula for the binomial coefficient:

$$\begin{aligned} \binom{n}{2} &= 66 \\ \frac{(n)(n-1)}{2} &= 66 \\ n^2 - n &= 132 \\ n^2 - n - 132 &= 0 \\ (n-12)(n+11) &= 0 \end{aligned}$$

   Thus either $n = 12$ or $n = -11$. We want the positive solution, so the answer is $\boxed{12}$.

5. Being the practical joker he is, Fred has changed the magic words used to open the Marauders Map. Help George figure out the new spell using the code below, and remember that magic always begins with the end.

HMZ KVH TIV ZHB SZR I

   $\boxed{SNAPESGREASYHAIR}$ In this cipher, known as the "Atbash cipher", the letter A is replaced by Z, B is replaced by Y, and so on. The full list of letter replacements is:

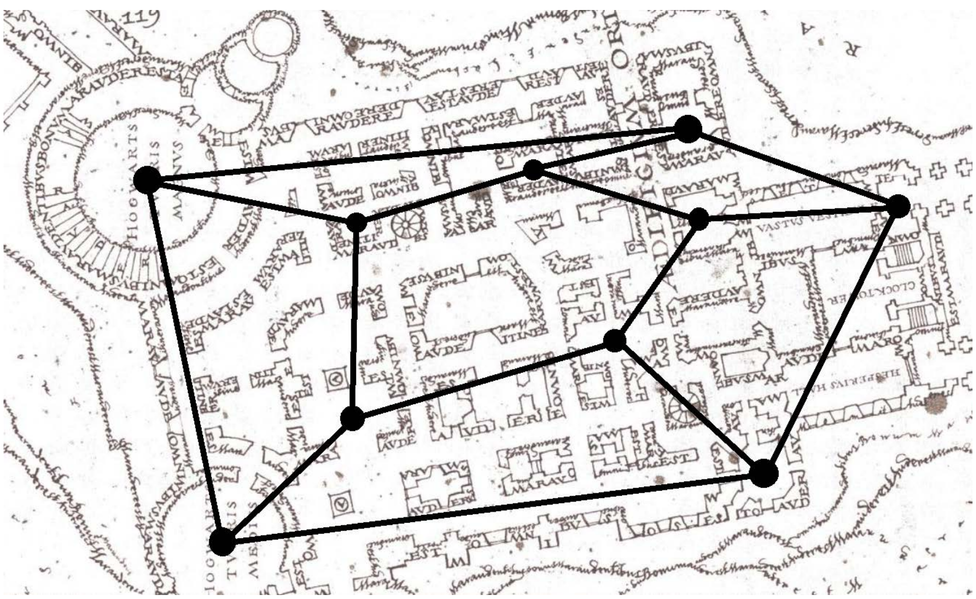
A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
Z	Y	X	W	V	U	T	S	R	Q	P	O	N	M	L	K	J	I	H	G	F	E	D	C	B	A

   Using this, we can decipher the message to receive "SNA PES GRE ASY HAI R".

6. With the help of the house elves, Fred and George are trying to smuggle Ten-Second Pimple Vanishers (TSPVs) between classrooms at Hogwarts. With the Yule Ball quickly approaching, demand for TSPVs has skyrocketed, as students want to look their best for the dance.

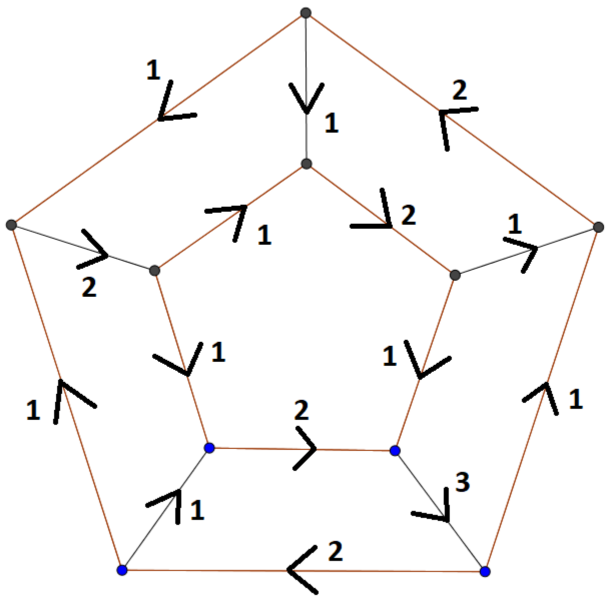
   The Marauder's Map has revealed secret passageways between certain classrooms, marked by lines on the map, but each passageway is only big enough for the house elves to travel in one direction at a time. To maximize the efficient sale of their TSPVs, the total number of TSPVs entering each classroom must be the same as the number of TSPVs exiting the classroom, and each passageway must have at least one TSPV traveling through it.

   Using the diagram on the Marauder's Map below, what is the least number of TSPVs required to satisfy these rules?



   $\boxed{22}$ We will call classrooms "vertices" and lines "edges". First, since each edge has to have at least 1 TSPV, each vertex must have at least 2 TSPVs entering it. If each of the 10 vertices had exactly 2 TSPVs entering it, there would be a total of $10 \cdot 2 = 20$ TSPVs. However, this is not possible; the next best is to have one vertex with 3 TSPVs going in and 3 coming out, while the rest remain 2 in and 2 out. This gives $\boxed{22}$ TSPVs.

Here is one example of a solution using 22 TSPVs:



7. As the final challenge in the Tri-Math Tournament, students must navigate a maze and reach the center in the fastest time. However, this year a record 25 students participated in the tournament, and due to safety concerns, only 5 at a time can be allowed into the maze for a race. Additionally, there are no stopwatches at Hogwarts, so the students times can only be measured relative to each other. What is the least number of races required to determine the fastest, second fastest, and third fastest students?

   $\boxed{7}$ First, group the students into 5 groups of 5 students each and race them against each other:

Races

	1	2	3	4	5
1^{st}	X	X	X	X	X
2^{nd}	X	X	X	X	X
3^{rd}	X	X	X	X	X
4^{th}	X	X	X	X	X
5^{th}	X	X	X	X	X

Each X represents a student. After these initial 5 races, we can eliminate all students who placed 4th or 5th, since there is no way they could be in the top 3 overall students (as they have already lost to at least 3 other students). This leaves us with 15 students:

Races

	1	2	3	4	5
1st	X	X	X	X	X
2nd	X	X	X	X	X
3rd	X	X	X	X	X
4th	~~X~~	~~X~~	~~X~~	~~X~~	~~X~~
5th	~~X~~	~~X~~	~~X~~	~~X~~	~~X~~

Now, for the 6th race, race the first place students from each of the five races and order them. The first place student in this race must be first overall, so we do not need to include this student in any further races. Additionally, the fourth and fifth place students in this race can be eliminated, but since the second and third place students from these students' initial races are slower than them, they can also be eliminated:

Races

1st overall	1	2	3	4	5
1st	X	X	X	~~X~~	~~X~~
2nd	X	X	X	~~X~~	~~X~~
3rd	X	X	X	~~X~~	~~X~~
4th	~~X~~	~~X~~	~~X~~	~~X~~	~~X~~
5th	~~X~~	~~X~~	~~X~~	~~X~~	~~X~~

However, we can use the information from the 6th race to eliminate 3 more students. In the third place student's initial race, the students who placed 2nd and 3rd can be at best 4th and 5th overall, respectively, so they can be eliminated. In the second place student's initial race, the student who placed 3rd can be at best 4th overall, and so can be eliminated, leaving 5 remaining students:

Races

1st overall	1	2	3	4	5
1^{st}	X	X	X	~~X~~	~~X~~
2^{nd}	X	X	~~X~~	~~X~~	~~X~~
3^{rd}	X	~~X~~	~~X~~	~~X~~	~~X~~
4^{th}	~~X~~	~~X~~	~~X~~	~~X~~	~~X~~
5^{th}	~~X~~	~~X~~	~~X~~	~~X~~	~~X~~

Race these five students; the first and second place students in this race will be the second and third place students overall. This is a total of $\boxed{7}$ races.