Diophantine Equations

Warm-Up

Math Circle Competition Team

January 15th, 2017

1. **(2015 AMC 12A)** Integers x and y with $x > y > 0$ satisfy $x + y + xy = 80$. Find x.

2. Use the Euclidean algorithm to find $\gcd(3456, 246)$.

3. **(2006 AMC 12A)** Two farmers agree that pigs are worth 300 dollars and that goats are worth 210 dollars. When one farmer owes the other money, he pays the debt in pigs or goats, with "change" received in the form of goats or pigs as necessary. (For example, a 390 dollar debt could be paid with two pigs, with one goat received in change.) What is the amount of the smallest positive debt that can be resolved in this way?

4. **(2015 AMC 10A)** Consider the set of all fractions $\frac{x}{y}$, where x and y are relatively prime positive integers. How many of these fractions have the property that if both numerator and denominator are increased by 1, the value of the fraction is increased by 10%?

5. **(1984 AHSME)** Find the number of triples (a, b, c) of positive integers which satisfy the simultaneous equations

$$ab + bc = 44$$

$$ac + bc = 23$$

Diophantine Equations

In-Class Problems

Math Circle Competition Team

January 15th, 2017

1. **(2000 AIME II)** A point whose coordinates are both integers is called a lattice point. How many lattice points lie on the hyperbola $x^2 - y^2 = 2000^2$?

2. **(2015 AIME I)** There is a prime number p such that $16p + 1$ is the cube of a positive integer. Find p.

3. **(1993 AIME)** How many ordered four-tuples of integers (a, b, c, d) with $0 < a < b < c < d < 500$ satisfy $a + d = b + c$ and $bc - ad = 93$?

4. **(1985 AIME)** The numbers in the sequence 101, 104, 109, 116,... are of the form $a_n = 100 + n^2$, where $n = 1, 2, 3, \ldots$ For each n, let d_n be the greatest common divisor of a_n and a_{n+1}. Find the maximum value of d_n as n ranges through the positive integers.

Diophantine Equations

Take-Home Problem Set

Math Circle Competition Team

Week of January 15th, 2017

1. **(2005 AIME II)** Let $P(x)$ be a polynomial with integer coefficients that satisfies $P(17) = 10$ and $P(24) = 17$. Given that $P(n) = n + 3$ has two distinct integer solutions n_1 and n_2, find the product $n_1 \cdot n_2$.

2. **(2008 AIME II)** Find the largest integer n satisfying the following conditions:

   - n^2 can be expressed as the difference of two consecutive cubes;
   - $2n + 79$ is a perfect square.

   Note: It may help to research Pell equations, a special type of Diophantine equation.