Diophantine Equations

Warm-Up: Solutions

Math Circle Competition Team

January 15th, 2017

1. **(2015 AMC 12A)** Integers x and y with $x > y > 0$ satisfy $x + y + xy = 80$. Find x.

   $\boxed{26}$ First, add 1 to both sides to receive $xy + x + y + 1 = 81$. Then we can factor this as $x(y + 1) + y + 1 = (x + 1)(y + 1) = 81$. Since x and y have to be integers, we have that $(x + 1)$ and $(y + 1)$ must be either 81 and 1, 27 and 3, or 9 and 9. Since $x \neq y$, the third option is eliminated, and since both are greater than 0, the first option is eliminated. Thus, since x is the larger value, we have $x + 1 = 27$ so that $x = \boxed{26}$.

2. Use the Euclidean algorithm to find $\gcd(3456, 246)$.

   $\boxed{6}$ We have $3456 = 246 \cdot 14 + 12$, then $246 = 12 \cdot 20 + 6$, and finally $12 = 6 \cdot 2 + 0$. Thus our gcd is $\boxed{6}$.

3. **(2006 AMC 12A)** Two farmers agree that pigs are worth 300 dollars and that goats are worth 210 dollars. When one farmer owes the other money, he pays the debt in pigs or goats, with "change" received in the form of goats or pigs as necessary. (For example, a 390 dollar debt could be paid with two pigs, with one goat received in change.) What is the amount of the smallest positive debt that can be resolved in this way?

   $\boxed{30}$ This is equivalent to the Diophantine equation $300x + 210y = z$, where x is the number of pigs and y is the number of goats. Then Bezout's Lemma tells us that the smallest possible integer value for z is $\gcd(300, 210)$, which is $\boxed{30}$.

4. **(2015 AMC 10A)** Consider the set of all fractions $\frac{x}{y}$, where x and y are relatively prime positive integers. How many of these fractions have the property that if both numerator and denominator are increased by 1, the value of the fraction is increased by 10%?

   $\boxed{1}$ In equation form, we wish to find x and y so that $\frac{x+1}{y+1} = 1.1 \cdot \frac{x}{y} = \frac{11x}{10y}$. Cross multiplying yields $10y(x + 1) = 11x(y + 1)$, or $10xy + 10y = 11xy + 11x$. Simplifying, we have $xy + 11x - 10y = 0$. We would like to factor this, so subtract 110 from both sides to receive $xy + 11x - 10y - 110 = (x - 10)(y + 11) = -110 = -2 \cdot 5 \cdot 11$. In other words, $(10 - x)(y + 11) = 2 \cdot 5 \cdot 11$. We now have a number of possibilities for $(10 - x)$ and $(y + 11)$: $(2, 55)$, $(55, 2)$, $(5, 22)$, $(22, 5)$, $(11, 10)$, or $(10, 11)$. First, since $(10 - x)$ must remain positive, we can eliminate $(55, 2)$, $(22, 5)$, $(11, 10)$, and $(10, 11)$, leaving only $(2, 55)$ and $(5, 22)$. But if $(10 - x) = 2$ and $y + 11 = 55$, this implies that

$x = 8$ and $y = 44$, which are not relatively prime. Thus our only choice is $(5, 22)$, which yields $x = 5$ and $y = 11$; these are relatively prime. Thus our answer is $\boxed{1}$.

5. **(1984 AHSME)** Find the number of triples (a, b, c) of positive integers which satisfy the simultaneous equations

$$ab + bc = 44$$

$$ac + bc = 23$$

$\boxed{2}$ Factoring, we have $b(a + c) = 44$ and $c(a + b) = 23$. Furthermore, if we subtract the second equation from the first equation, we receive $ab - ac = a(b - c) = 21$. Since $c(a + b) = 23$, and 23 is prime, we must have that c and $a + b$ equal 23 and 1; since a and b must be positive, we cannot have $a + b = 1$, so $c = 1$ and $a + b = 23$. Our other equations now reduce to $b(a + 1) = 44$ and $a(b - 1) = 21$. Trying factor pairs subject to $a + b = 23$ leaves two possibilities: $a = 1$ and $b = 21$, or $a = 21$ and $b = 2$. Thus our answer is $\boxed{2}$.

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In-Class Problems: Solutions

Math Circle Competition Team

January 15th, 2017

1. **(2000 AIME II)** A point whose coordinates are both integers is called a lattice point. How many lattice points lie on the hyperbola $x^2 - y^2 = 2000^2$?

   $\boxed{98}$ We have $x^2 - y^2 = (x+y)(x-y) = 2000^2 = (2^4 \cdot 5^3)^2 = 2^8 \cdot 5^6$, so we wish to find the number of integer solutions to $(x-y)(x+y) = 2^8 \cdot 5^6$. If x were even and y odd, we would have $(x-y)$ and $(x+y)$ both odd, so 2 could not be in the prime factorization of their product. Thus either both x and y are even or both are odd; regardless, each of $(x-y)$ and $(x+y)$ must be even. This fixed two of our 2's, leaving the factors $2^6 \cdot 5^6$ to be distributed between $(x+y)$ and $(x-y)$. Once we have picked the factors of $(x+y)$, $(x-y)$ is fixed, so we'll focus on the possibilities for $(x+y)$. For factors of 2, we can pick $2^0, 2^1, \ldots, 2^6$ and for factors of 5, we can pick $5^0, 5^1, \ldots, 5^6$. Thus we have $7 \cdot 7 = 49$ choices for $(x+y)$. However, $(x+y)$ could be negative, yielding another 49 lattice points, for a total of $49 + 49 = \boxed{98}$.

2. **(2015 AIME I)** There is a prime number p such that $16p+1$ is the cube of a positive integer. Find p.

   $\boxed{307}$ We seek an x so that $x^3 = 16p + 1$. First, since $16p + 1$ is odd, x must be odd. Next we have $x^3 - 1 = 16p$, so that $(x-1)(x^2+x+1) = 16p$. Since x is odd, we must have that $(x-1)$ is even and (x^2+x+1) is odd. Then 16 divides $(x-1)$, but if $(x-1)$ were any larger multiple of 16, then p would not be prime. Thus $x - 1 = 16$ and $x = 17$. Now we just solve for p: $16(17^2 + 17 + 1) = 16p$ so that $p = \boxed{307}$.

3. **(1993 AIME)** How many ordered four-tuples of integers (a, b, c, d) with $0 < a < b < c < d < 500$ satisfy $a + d = b + c$ and $bc - ad = 93$?

   $\boxed{870}$ (Solution taken from aops.com, 1993 AIME Problem 4) Let $k = a + d = b + c$ so $d = k - a, b = k - c$. It follows that $(k-c)c - a(k-a) = (a-c)(a+c-k) = (c-a)(d-c) = 93$. Hence $(c-a, d-c) = (1, 93), (3, 31), (31, 3), (93, 1)$.

   Solve them in tems of c to get $(a,b,c,d) = (c-93, c-92, c, c+1)$, $(c-31, c-28, c, c+3)$, $(c-1, c+92, c, c+93)$, $(c-3, c+28, c, c+31)$. The last two solutions don't follow $a < b < c < d$, so we only need to consider the first two solutions.

   The first solution gives us $c - 93 \geq 1$ and $c + 1 \leq 499 \implies 94 \leq c \leq 498$, and the second one gives us $32 \leq c \leq 496$.

   So the total number of such four-tuples is $405 + 465 = \boxed{870}$.

4. **(1985 AIME)** The numbers in the sequence 101, 104, 109, 116,... are of the form $a_n = 100+n^2$, where $n = 1, 2, 3, \ldots$ For each n, let d_n be the greatest common divisor of a_n and a_{n+1}. Find the maximum value of d_n as n ranges through the positive integers.

   $\boxed{401}$ (Solution taken from aops.com, 1985 AIME Problem 13) We know that $a_n = 100 + n^2$ and $a_{n+1} = 100 + (n+1)^2 = 100 + n^2 + 2n + 1$. Since we want to find the GCD of a_n and a_{n+1}, we can use the Euclidean algorithm:

$$a_{n+1} - a_n = 2n + 1$$

   Now, the question is to find the GCD of $2n+1$ and $100+n^2$. We subtract $2n+1$ 100 times from $100+n^2$. This leaves us with n^2-200n. We want this to equal 0, so solving for n gives us $n = 200$. The last remainder is 0, thus $200 * 2 + 1 = \boxed{401}$ is our GCD.

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Take-Home Problem Set: Solutions

Math Circle Competition Team

Week of January 15th, 2017

1. **(2005 AIME II)** Let $P(x)$ be a polynomial with integer coefficients that satisfies $P(17) = 10$ and $P(24) = 17$. Given that $P(n) = n + 3$ has two distinct integer solutions n_1 and n_2, find the product $n_1 \cdot n_2$.

   $\boxed{418}$ (Solution taken from aops.com, 2005 AIME II Problem 13) We define $Q(x) = P(x)-x+7$, noting that it has roots at 17 and 24. Hence $P(x)-x+7 = A(x-17)(x-24)$. In particular, this means that $P(x) - x - 3 = A(x - 17)(x - 24) - 10$. Therefore, $x = n_1, n_2$ satisfy $A(x-17)(x-24) = 10$, where A, $(x-17)$, and $(x-24)$ are integers. This cannot occur if $x \leq 17$ or $x \geq 24$ because the product $(x - 17)(x - 24)$ will either be too large or not be a divisor of 10. We find that $x = 19$ and $x = 22$ are the only values that allow $(x - 17)(x - 24)$ to be a factor of 10. Hence the answer is $19 \cdot 22 = \boxed{418}$.

2. **(2008 AIME II)** Find the largest integer n satisfying the following conditions:

   - n^2 can be expressed as the difference of two consecutive cubes;
   - $2n + 79$ is a perfect square.

   Note: It may help to research Pell equations, a special type of Diophantine equation.

   $\boxed{181}$ (Solution taken from aops.com, 2008 AIME II Problem 15) Suppose that the consecutive cubes are m and $m + 1$. We can use completing the square and the first condition to get:

   $$(2n)^2 - 3(2m + 1)^2 = 1 \equiv a^2 - 3b^2$$

   where a and b are non-negative integers. Now this is a Pell equation, with solutions in the form $(2 + \sqrt{3})^k = a_k + \sqrt{3}b_k$, $k = 0, 1, 2, 3, ...$ However, a is even and b is odd. It can be seen that the parity of a and b switch each time (by induction). Hence all solutions to the first condition are in the form:

   $$(2 + \sqrt{3})^{2k+1} = a_k + \sqrt{3}b_k$$

   where $k = 0, 1, 2, ...$ So we can obtain the following: $(k, a_k = 2n) = (0, 2), (1, 26), (2, 362), (3, 5042)$. It is an AIME problem so it is implicit that $n < 1000$, so $2n < 2000$. Then, by induction, a_n is strictly increasing. Checking $2n = 362 \implies n = 181$ in the second condition works (we know b_k is odd so we don't need to find m). So our answer is $\boxed{181}$.