Combinatorics Potpourri: Facts

Math Circle Competition Team

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- **Relatively Prime.** Two positive integers m and n are said to be *relatively prime* if they share no common divisors greater than one; in other words, if their greatest common divisor is 1. This is written as $\gcd(m, n) = 1$. For example, $21 = 7 \cdot 3$ and $22 = 11 \cdot 2$ are relatively prime, as their gcd is 1.

- **Chicken McNugget Theorem.** Suppose that at McDonalds, you can buy Chicken McNuggets in packs of 20 or 9. If you cannot eat or take away any nuggets, what is the largest number of nuggets that you cannot purchase? The answer is 151, which is $20 \cdot 9 - 20 - 9$. This leads to the Chicken McNugget Theorem.

  The theorem states that for any two relatively prime positive integers m and n, the greatest integer that cannot be written in the form $am + bn$ for nonnegative a and b is

  $$mn - m - n.$$

- **Stars and Bars.** In how many ways can we distribute n indistinguishable objects (represented below as stars) into k bins? In the case where $n = 7$ and $k = 3$, we have seven stars:

  $$\star \star \star \star \star \star \star$$

  and we wish to separate them into 3 bins (represented by lines). For example, we might separate them as below:

  $$\star \star \star \star \mid \star \mid \star \star$$

  with one bin of 4 stars, one bin of 1 star, and one bin of 2 stars. We can put a separator in any gap between two stars (there are $n - 1$ such gaps) and we will place $k - 1$ separators, as this results in exactly k bins. Thus the total number of ways to distribute k objects into n bins (when a bin must contain at least one object) is

  $$\binom{n-1}{k-1}.$$

  If, however, a bin can remain empty (contain 0 objects), the total number of ways is

  $$\binom{n+k-1}{k-1}.$$

- **Pigeonhole Principle.** If n pigeons are placed in m holes, where $m < n$, then one hole must contain more than one pigeon.