Crossing Number of Complete Bipartite Graphs

Amanda Phillips, Kalina Namikas, Sarah Li

MATH CIRCLE

LSU Louisiana State University

Introduction

During World War II, a Hungarian mathematician, named Pál Turán, was sent to a brick factory. He was forced to move wagonfuls of bricks from the factory to a storage warehouse using wagon tracks set up around the factory. Turán noticed that where the tracks intersected, the cart jumped, and all of the bricks fell out. He wondered what the fewest number of intersections needed to connect every factory to every warehouse was.

In 1952, Polish mathematician Kazimierz Zarankiewicz attempted this problem and used his previous experience in topology to derive a formula for the minimum number of intersections given a certain number of vertices. Later, British mathematician Richard Guy found an error in Zarankiewicz's proof, and the proof of his formula is still unresolved today.

We are attempting to find a formula for the minimum crossing number of complete bipartite graphs without first being influenced by Zarankiewicz's and Guy's ideas.

Background

- A graph is a set of dots, called **vertices**, and lines connecting them, called **edges**.
- Vertices connected by an edge are called **adjacent**.
- Graphs can be moved around and are still considered **isomorphic**, as long as the same vertices are adjacent in the two graphs.
- **Planar Graphs** are graphs that can be drawn in a plane with no edges intersecting.
- **Non-Planar Graphs** are graphs that cannot be drawn in a plane without any intersecting edges.
- **Bipartite graphs** are graphs that have two sets of vertices, where no two vertices in the same set are adjacent.
- **Complete bipartite graphs** are bipartite graphs in which every vertex in one set is adjacent to every vertex in the other set.

A $K_{3,3}$ graph. No vertex in the same set is adjacent, and every vertex is adjacent to every vertex in the other set.

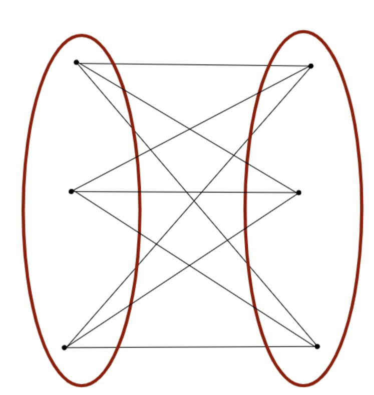


Fig. 1: A Complete Bipartite Graph

- The **crossing number** (denoted $\mathrm{Cr}(K_{m,n})$)of a graph is the least number of edges that must intersect in any isomorphism of the original graph.

Strategy

The general method for finding the minimum number of crossings involves placing both sets of points in lines and lining them up perpendicular to each other to prevent excessive overlap. While there are other methods, such as lining up both sets parallel to each other, this method is not as optimal, seeing as this graph outputs seven crossings while ours only outputs one.

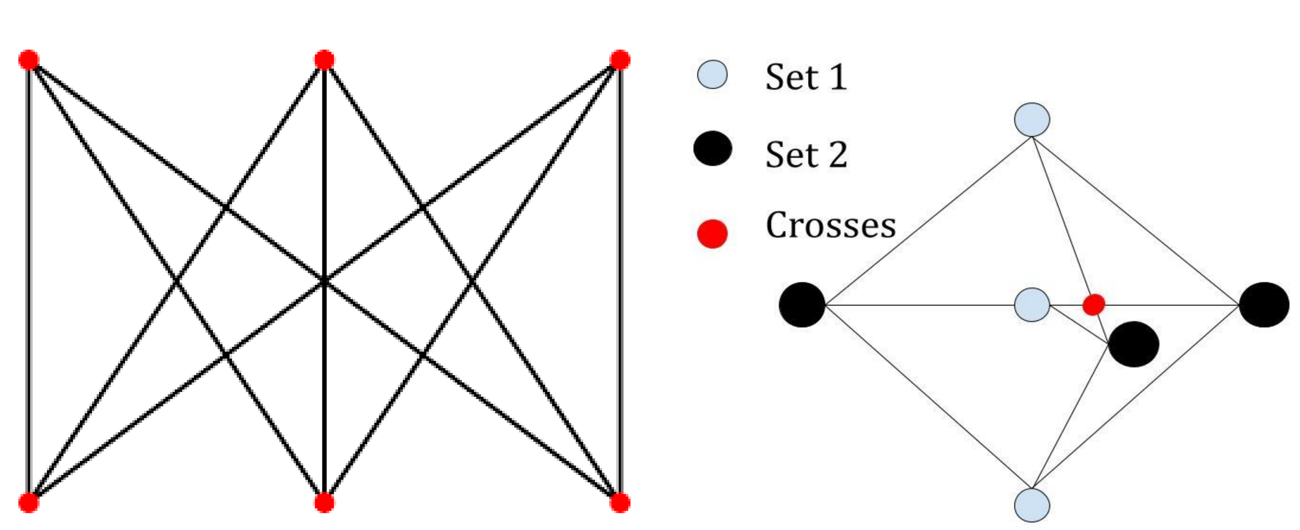


Fig. 2: $\mathrm{Cr}(K_{3,3}) = 1$

Determining An Equation

Once we found the optimal strategy for finding crossing numbers by hand, we created a table. We then started looking for patterns between the even and odd numbers where either m or n was constant.

	n											
m	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**
3	1	2	4	6	9	12	16	20	25	30	36	42
4	2	4	8	12	18	24	32	40	50	60	72	84
5	4	8	16	24	36	48	64	80	100	120	144	168
...												
12	30	60	120	180	270	360	480	600	750	900	1080	1260
13	36	72	144	216	324	432	576	720	900	1080	1296	1512
14	42	84	168	252	378	504	672	840	1050	1260	1512	1764

Crossing Number of Complete Bipartite Graphs

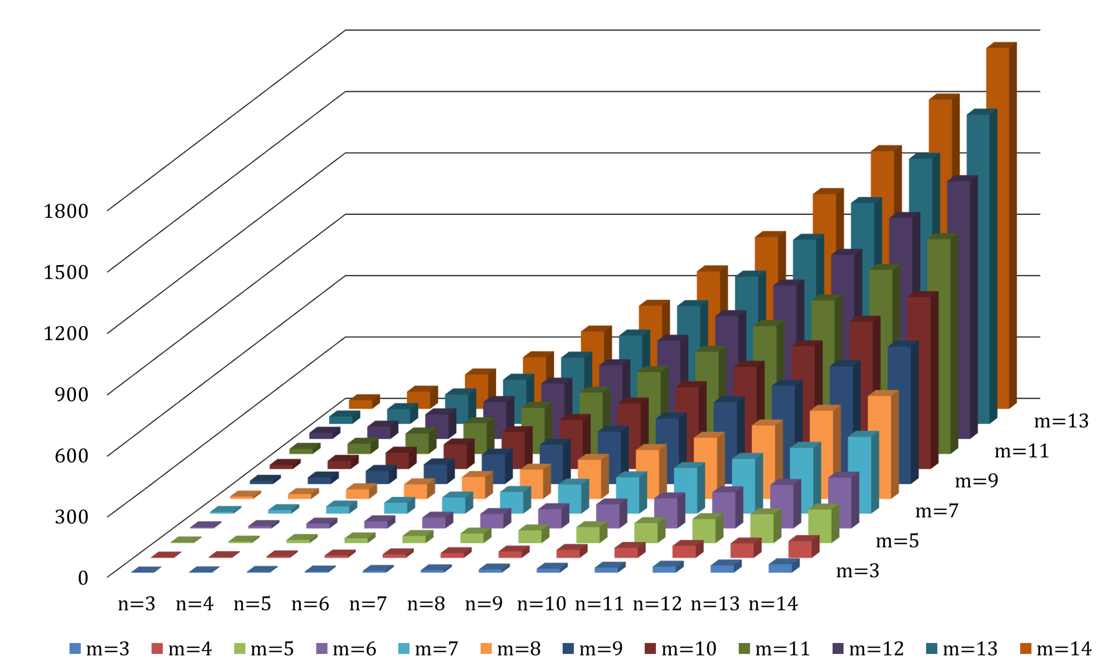


Fig 3: Graph of the table above

We found formulas for those patterns, then took the similarities and used them to build three generalized formulas:

Case 1: Both m and n are even

$$\frac{n(n-2)}{4} \cdot \frac{m(m-2)}{4}$$

Case 2: Both m and n are odd

$$\left(\frac{n(n-2)}{2}\right)^2 \cdot \left(\frac{m(m-2)}{2}\right)^2$$

Case 3: n is even and m is odd

$$\frac{n(n-2)}{4} \cdot \left(\frac{m(m-2)}{2}\right)^2$$

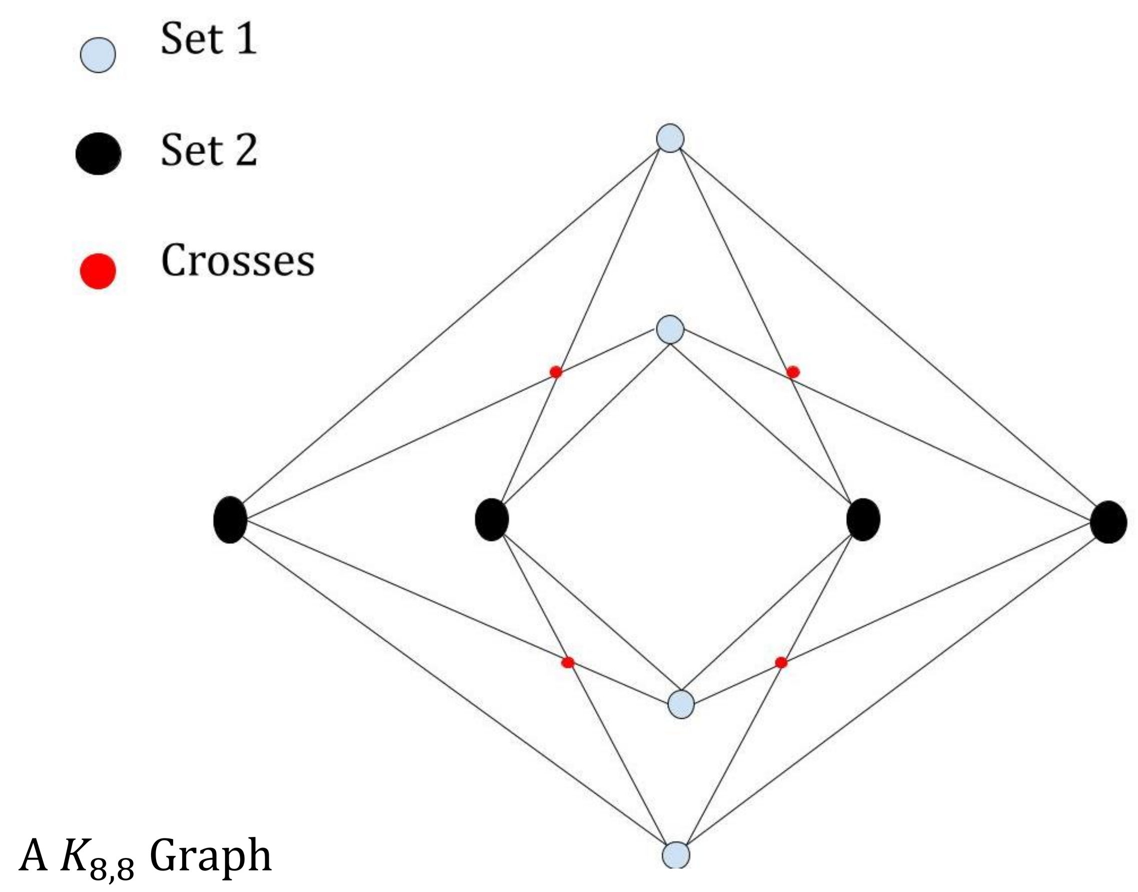
When checked against every value in the table, the equations held true. However, we needed three different equations. Next, we tried to simplify them to match Zarankiewicz's formula. His formula was:

$$\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor$$

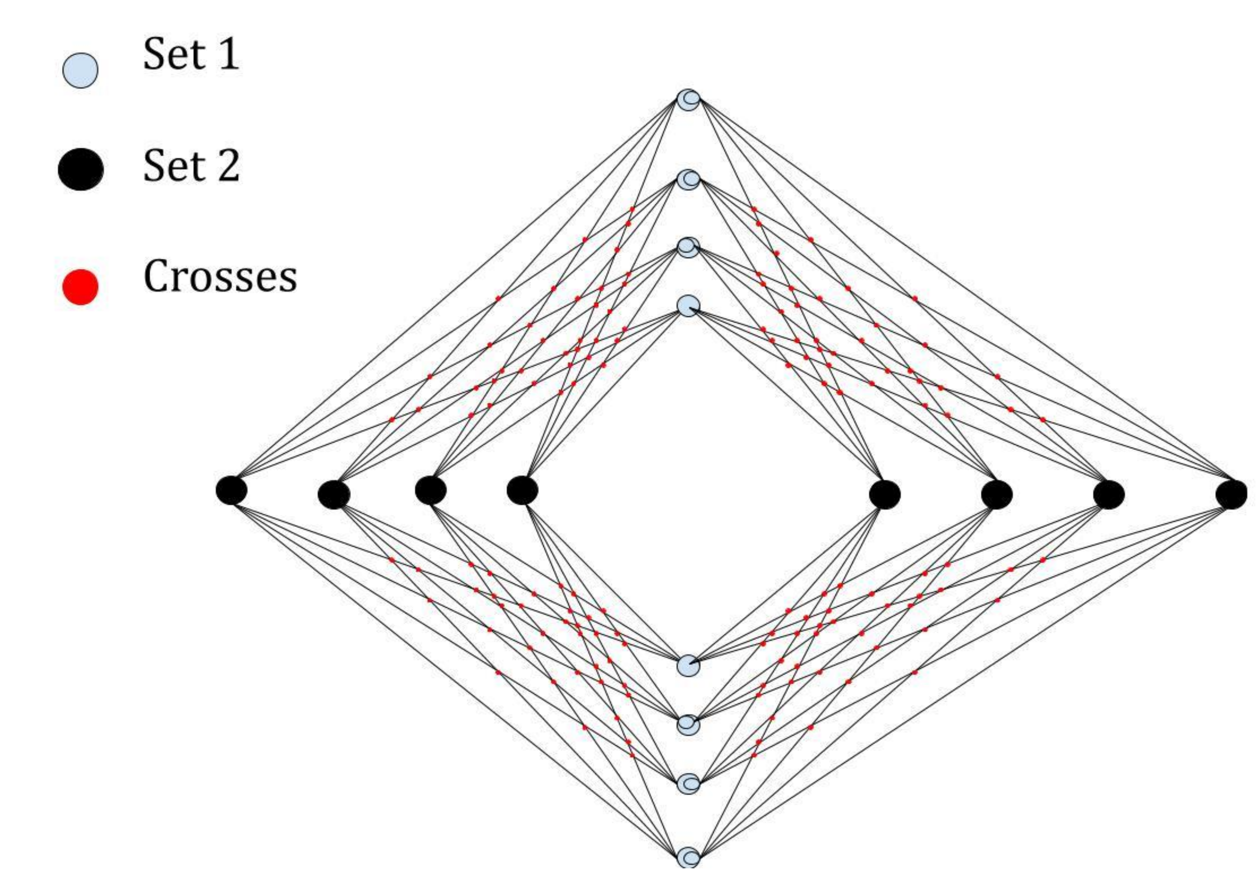
One possible reason why we had three equations and Zarankiewicz only required one was because of his use of the floor function. However, we used a ceiling function instead of a floor function to get the same result. After some factoring, the sections of our equations that accounted for even or odd numbers were the same. Thus, we simplified our three equations that were specific to even or odd cases down to one equation:

$$Cr(K_{m,n}) = \left\lceil \frac{n^2-2n}{4} \right\rceil \left\lceil \frac{m^2-2m}{4} \right\rceil$$

A $K_{4,4}$ Graph



A $K_{8,8}$ Graph



Conclusion

Although our formula still remains unproved, it provides a theoretical lower bound on the crossing number of complete bipartite graphs. We do not have a formal proof, but we do have the empirical evidence that supports our formula. Our formula is based upon the strategy for getting the lowest crossing number, so if we could justify our strategy then we could prove our formula. Further work to be done on this topic would include finding a proof of our formula or trying to determine if our formula and Zarankiewicz's are the same.

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